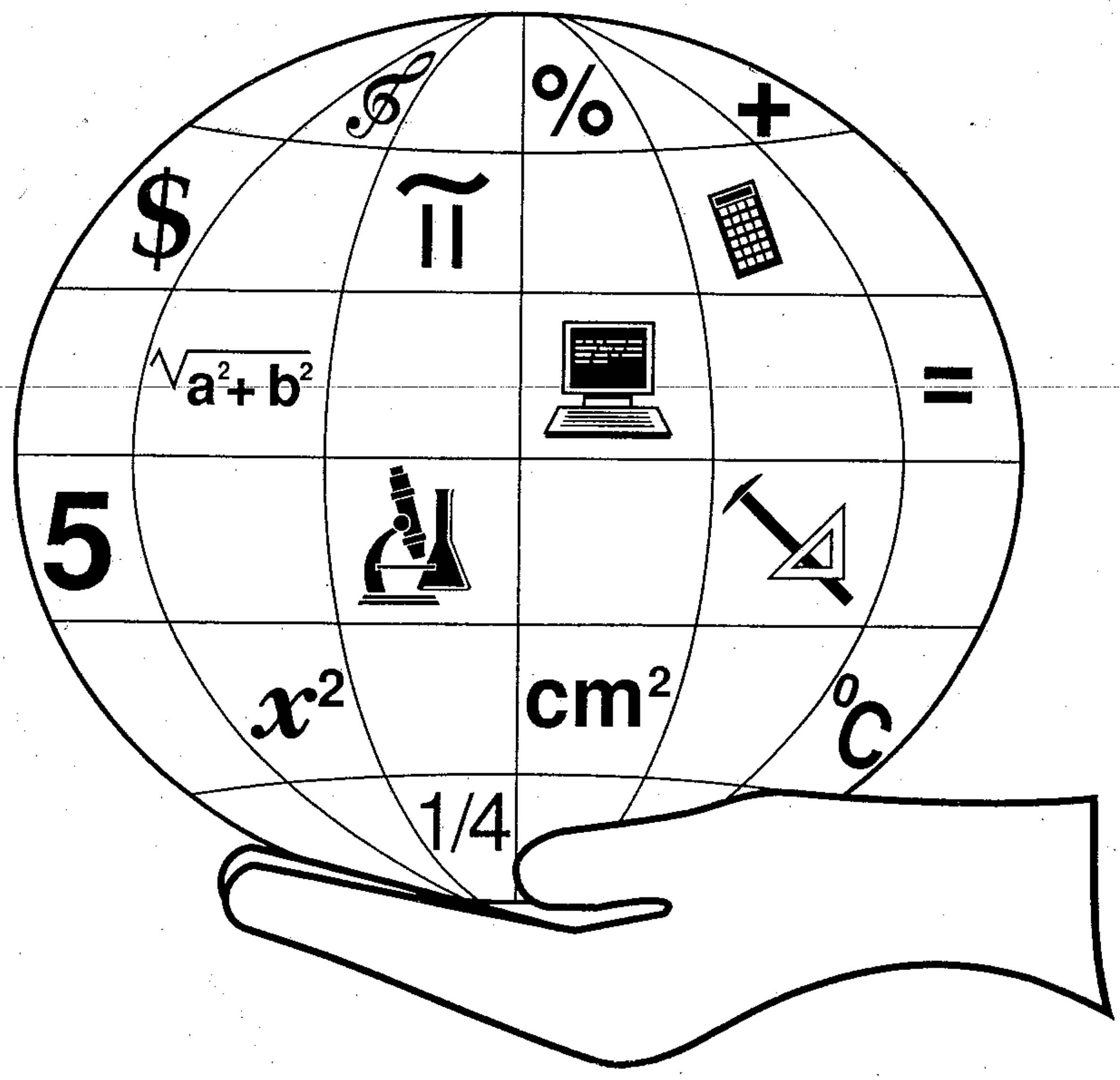
Saskatchewan
Learning

Mathematics A30, B30, C30

A Curriculum Guide for the Secondary Level



September 1996

Mathematics A 30, B 30, C 30

A Curriculum Guide for the Secondary Level

Saskatchewan Education
September 1996
ISBN 0-921291-37-X

Acknowledgements

Saskatchewan Education gratefully acknowledges the professional contributions and advice given by the following Committee members:

Reference Committee Members (1993 - 1996):

Dr. David Bale
Faculty of Education
University of Regina

Dr. James Beamer (retired 1995)
College of Education
University of Saskatchewan

Mr. Eric Hamm
Teacher
Northern Lights S.D. No. 113

Mr. Ken Norris
Teacher
Carlton Comprehensive High School Board

Ms Gale Russell
Teacher
Last Mountain S.D. No. 29

Dr. Garth Thomas, Professor (retired 1995)
Department of Mathematics
University of Saskatchewan

Ms Darla Wall Carignan
Teacher
Regina S.D. No. 4

Provincial Curriculum Advisory Committee Members (1988 - 1993):

Dr. James Beamer
Professor, Mathematics Education
University of Saskatchewan

Mr. Harold Flett
Principal (Community Education)
Cumberland House

Mr. Mike Fulton
Director of Education
Indian Head S.D. No. 19

Ms Lillian Gauthier
Teacher (Elementary; OMLO)
Saskatoon S.D. No. 13

Dr. Denis Hanson
Professor, Department of Mathematics and Statistics
University of Regina

Mr. Bob Kallio
Saskatchewan School Trustees Association
Dinsmore, Saskatchewan

Dr. Ed Klopoushak
Professor, Mathematics Education
University of Regina

Mr. Thom Koroluk
Principal (Elementary)
Yorkton S.D. No. 93

Ms Sharon Kvinlaug
Instructor, SIAST
Prince Albert

Mr. Laurence Owen
Teacher (Secondary)
Saskatoon S.D. No. 13

Ms Verda Petry
Teacher (Secondary)
Regina S.D. No. 4

Mr. Derek Punshon
Principal (K-12)
Wood River S.D. No. 70

Ms Mary Reeves
Principal
Regina R.C.S.S.D. No. 81

Ms. Ruth Taylor
Saskatchewan School Trustees Association
Saskatoon, Saskatchewan

Dr. Garth Thomas
Professor, Department of Mathematics
University of Saskatchewan

Ms Heather Wright
Vice Principal
Regina S.D. No. 4

Ms Judy Wall
Vice Principal (Special Education)
Regina S.D. No. 4

Saskatchewan Education wishes to thank many others who contributed to the development of the Curriculum Guide:

- pilot teachers;
- the Mathematics Program Team;
- Burt Thiessen, contracted consultant;
- in-house consultants; and,
- various field personnel.

This document was completed under the direction of the Mathematics, Science, and Technology Unit at Saskatchewan Education.

Foreword

Much of the foundation for curriculum renewal in Saskatchewan schools is based on *Directions* (1984). The excitement surrounding the recommendations for Core Curriculum developments will continue to build as curricula are designed and implemented to prepare students for the 21st century.

Mathematics, as one of the Required Areas of Study, incorporates the Common Essential Learnings and the Adaptive Dimension. In addition, other Core Curriculum initiatives such as Gender Equity, Indian and Métis content and perspectives, and Resource-based Learning are also addressed.

As we strive to achieve the goals of mathematics education in Saskatchewan schools, much collaboration and cooperation among individuals and groups will be required. Mathematics teachers are a key part of the process.

Table of Contents

Introduction

Philosophy, Aim, and Goals

The philosophy of mathematics education in Saskatchewan is reflected in the program aim and goals. In addition, the philosophy is closely related to the concept of Core Curriculum based on *Directions* (1984) and the **Goals of Education for Saskatchewan**.

The aim of the mathematics program is to graduate numerate individuals who value mathematics and appreciate its role in society. This is an attempt to enable students to cope confidently and competently with everyday situations that demand the use of mathematical concepts. Specifically, this means interpreting quantitative information, estimating, performing calculations mentally, and developing an intuitive knowledge of measurement and spatial relationships. In addition, the mathematics program is intended to stimulate the spirit of inquiry by developing a variety of problem solving skills and abilities. Lastly, there is a need to make effective use of technology where it is most appropriate.

The curriculum goals are intended to provide for students the mathematical preparation essential to:

- function as consumers and workers; that is, to develop the skills and knowledge of concepts necessary to meet the needs of the average worker and consumer. This can be accomplished by developing an understanding of the relationship between problem solving in the real-world and the mathematics taught in schools;
- function as informed responsible citizens; that is, to develop the ability to analyze and interpret quantitative information.
- obtain a liberal education; that is, to develop logical thinking skills, effective work habits and an appreciation of mathematics;
- become capable problem solvers; that is, to develop the desire, confidence and ability to solve problems;
- communicate mathematically; and,
- pursue further study in mathematics and mathematically related areas.

Emphasis is placed on how to compute, measure, estimate, and interpret mathematical data and when to apply these same skills and techniques. An understanding of why these processes apply is also stressed. The intent is to develop self-reliant, self-motivated, confident life-long learners.

Foundational Objectives

The Foundational Objectives describe the most important understandings and abilities which should be developed over the course of a unit or a year. They provide guidance to teachers in unit and yearly planning and should be within the range of abilities of the majority of students.

The Foundational Objectives form the basis for curriculum evaluation.

The Foundational Objectives are listed at the beginning of the sections dealing with the Mathematics A 30, B 30, C 30 Curriculum. Included with them, for awareness, are the General Outcomes of the Western Canadian Protocol, (1996).

Using the Curriculum Guide

Course Overview

The content of this course is broadly based and practical. The structure of the program includes seven mathematical strands.

i) Data Analysis and Consumer Mathematics
ii) Numbers and Operations
iii) Equations
iv) Algebra
v) Functions
vi) Geometry
vii) Trigonometry

Problem solving is an integral component of all strands, and is to be incorporated throughout the program. **The concept of function is stressed, as the understanding of many of the strands of mathematics depend upon knowledge of this concept.** Concepts are further supported by a number of learning objectives/skills, all of which emphasize and develop the Foundational Objectives of mathematics and the Common Essential Learnings. Mathematics A 30, B 30, and C 30 are intended to meet the needs of all students. These courses are flexible enough to recognize and encourage the development of options to meet the needs of students with both high and low achievement levels. Mathematics A 30, B 30 and C 30 are intended for students who have good ability in mathematics and who require mathematics in any future pursuits.

Any single resource will not cover all of the concepts and skills of this curriculum guide. Instead, a variety of resources/materials should be employed to select activities and content that coincide with student learning styles, individual teaching styles, and the philosophy of the curriculum.

The development and sequencing of the concepts should follow a logical progression using certain necessary principles to show the relationships between the concepts. Teachers should not restrict themselves to using just one instructional strategy when teaching a concept. Teachers might consider the intellectual aptitude of their students, what they already know about the concept, the nature of the concept, its significance in the structure of the other mathematical concepts, and the level of performance expected when choosing among the teaching strategies available.

There are many ways that the Adaptive Dimension can be incorporated into Mathematics. They include:

- **altering the method of instruction to meet individual needs;**
- **altering the setting so that students may benefit more fully from instruction;**
- **altering the pace of the lesson to ensure that students understand the concepts; and,**
- **altering the method in which students are required to respond to the teacher and/or to the instructional approach.**

It should be remembered that the less rigid the setting and the approach, the easier it is to adapt. Any method, or some combination of methods, is acceptable. **Suggested ideas are given for helping students to achieve various skills/objectives in this guide in the Instructional Notes and in the Adaptations column.** Additional ideas are provided in *Instructional Approaches: A Framework for Instructional Practice* (Saskatchewan Education, 1991) and *The Adaptive Dimension in Core Curriculum* (Saskatchewan Education, 1992).

Conceptual Overviews

The conceptual overviews for each of Mathematics 10, 20, A 30, B 30, and C 30 are listed in the section ***Aids for Planning***.

The sequencing of these concepts is at the discretion of the teacher. However, a logical order showing the relationships among the various concepts should be considered. For instance, several examples from consumer mathematics can be used to reinforce the concept of linear functions. In order to develop understanding, creative arrangement of the concepts is encouraged. The order in which the concepts are developed could be modified, or several concepts could be integrated.

It should be noted that all the learning objectives for grade 9 have not been included in the Secondary Scope and Sequence. Only those concepts that carry through to successive courses have been indicated.

Reference List for the Scope and Sequence

There are two versions of the "Scope and Sequence" in this guide. One "Scope and Sequence" is a listing of the Foundational Objectives and the supporting learning objectives for each grade, while the second "Scope and Sequence" shows the development of each of the strands over the course of the entire program.

Guide To Using Resource Materials

As was indicated earlier, no single resource matches the Mathematics curriculum. To facilitate a resource-based approach, **the use of a variety of resources instead of a single textbook is highly recommended.**

Teachers may find it necessary for each student to have a basal textbook that covers the majority of the content. If so, then other reference texts must be used. Some teachers may wish to have their students work in groups of two or three and use multiple recommended texts. In any case, the approach should coincide with student learning styles and individual teaching styles. It is also recommended that non-print materials such as software and videos be used in order to enhance the delivery of the course.

A resource-based learning approach requires long-term planning and coordination within a school or school division. In-school administrators, the teacher-librarian, and others need to take an active role to assist with this planning.

Instructional approaches which emphasize group work and develop independent learning abilities make it possible to utilize limited resources in a productive way.

Core Curriculum Components and Initiatives

Core Curriculum: Plans for Implementation (1987) defines the Core Curriculum as including seven Required Areas of Study, the Common Essential Learnings, the Adaptive Dimension, and Locally-Determined Options. Mathematics is one of the Required Areas of Study.

Core Curriculum initiatives also include Gender Equity, Indian and Métis Perspectives, and Resource-based Learning. These initiatives can be viewed as principles which guide the development of curricula as well as instruction in the classroom. The components and initiatives outlined in the following statements have been integrated throughout the curriculum.

Common Essential Learnings

Understanding the Common Essential Learnings: A Handbook for Teachers (1988) defines and expands on an understanding of these essential learnings. These may be considered the "New Basics".

Mathematics offers many opportunities for incorporating the Common Essential Learnings (C.E.L.s) into instruction. The purpose of this incorporation is to help students better understand mathematics and to prepare students for their future learning both within and outside of the K-12 educational system. The decision to focus on particular C.E.L.s within a lesson is guided by the needs and abilities of individual students and by the particular demands of mathematics. Throughout a unit, it is intended that each of the Common Essential Learnings will be developed to the extent possible.

It is important to incorporate the C.E.L.s in an authentic manner. For example, some areas of mathematics may offer many opportunities to develop the understandings, values, skills and processes related to a number of the Common Essential Learnings. The development of a particular C.E.L., however, may be limited by the nature of the subject matter under study.

It is intended that the Common Essential Learnings be developed and evaluated within subject areas. Therefore, Foundational Objectives for the C.E.L.s are included within the overviews of the strands in this guide. Because the Common Essential Learnings are not necessarily separate and discrete categories, it is anticipated that working toward the achievement of one Foundational Objective may contribute to the development of others. For example, many of the processes, skills, understandings and abilities required for the C.E.L.s of Communication, Numeracy, and Critical and Creative Thinking are also needed for the development of Technological Literacy.

Incorporating the Common Essential Learnings into instruction has implications for the assessment of student learning. A unit which has focused on developing the C.E.L.s of Communication and Critical and Creative Thinking should also reflect this focus when assessing student learning. Assessment should allow students to demonstrate their understanding of the important concepts in the unit and how these concepts are related to each other or to previous learning. Questions can be structured so that evidence or reasons must accompany student explanations. If students are encouraged to think critically and creatively throughout a unit, then the assessment for the unit should also require students to think critically and creatively.

It is anticipated that teachers will build from the suggestions in this guide and from their personal reflections in order to incorporate the Common Essential Learnings into mathematics.

For example, involving students in groups to solve realistic problems helps to develop Personal and Social Values and Skills. Similarly, realistic problems provide a medium to promote the important aspects of human communication: listening, speaking, reading and writing. Additionally, critical thinking can be developed in the mathematics program by providing students with an opportunity to evaluate statistical claims made in advertising and by asking "what if" questions in geometry. Numeracy is naturally developed throughout, especially in the interpretation of quantitative information and the use of probability, ratios, and proportions. All serve to assist students to cope confidently and competently with everyday situations. Having students actively using the calculator as a problem solving tool and applying computer spreadsheets to organize data and technological information develops their awareness of technology in an ever changing world. Independent Learning is fostered by encouraging students to investigate the applications, history, and further study of mathematics. In creating such opportunities and experiences, students will become capable, self-reliant, self-motivated, and life-long learners.

Resource-based Learning

Personnel, collections, facilities, and budgets in Saskatchewan school libraries vary a great deal, and the quality of school library programs and services is therefore not consistent. Possibilities for resource-based instruction are related to the level of administrative and staff commitment to developing well-staffed and well-equipped school libraries.

Resource-based teaching and learning is a means by which teachers can greatly assist the development of attitudes and abilities for independent life-long learning. Resource-based learning is student-centred. It offers students opportunities to choose, to explore, and to discover. Students who are encouraged to think critically in an environment rich in resources are well on their way to becoming autonomous learners.

It is important for the mathematics teacher to cooperate with library staff to integrate non-print, human, and print resources with classroom assignments. The classroom teacher plans in advance with library staff, and respects the library resource centre as an extension of the classroom and a place for active learning. The librarian selects materials for the collection based on reviews in professional journals and invites or encourages the input of classroom teachers. The teacher-librarian, if available, could assist with planning assignments, integrating appropriate resources, and teaching students the processes needed to find, use, and present information.

The library resource centre staff could support the mathematics curriculum by:

a) displaying curiosity, and modelling open-ended investigation, and problem-solving approaches;

b) demonstrating use of electronic networks and databases to link to sources that support mathematics interests;

c) organizing and circulating print and non-print resources which support the mathematics curriculum. Resources might include manipulatives, commercial games, videos, filmstrips and films, software, newspapers and magazines, reference books containing statistics and other numerical data, maps and globes, scale drawings, and measuring instruments;

d) maintaining a resource file of speakers and presenters in the community who can contribute their mathematics career experience to the classroom;

e) assisting the mathematics teacher to set up learning stations in the classroom or library using library resources;

f) cooperating with the mathematics teacher to teach students methods of library organization including computerized systems, and practical uses of indexing of all kinds;

g) providing resources for students at all levels of ability including exceptional children;

h) maintaining a collection of professional material on subjects of interest to mathematics teachers;

i) providing a link to information electronic databases and materials from other libraries, the central board office, universities, museums, governments, and industry;

j) providing enrichment materials which anticipate students' interests such as books of puzzles, mathematical games, material on crafts and hobbies using mathematical principles, magazines which deal with mathematics/science, and sports records; and,

k) providing interdisciplinary learning, to help students comprehend and anticipate the links between mathematics and other disciplines and areas of study.

l) helping to access Internet sites; for example, (http://fermat.math.uregina.ca/math_central/) has been developed at the University of Regina by the Mathematics and Statistics Department with cooperation from the Education Faculty for all K-12 teachers, student teachers, and students in the Province of Saskatchewan.

Instructional Approaches

It is necessary for teachers to use a broad range of instructional approaches to give students a chance to develop their understandings and abilities to investigate, to make sense of, and to construct meanings from new situations; to make and provide arguments for conjectures; and to use a flexible set of strategies to solve problems from both within and outside mathematics. In addition, greater opportunities can be provided for small-group work, independent learning, electronic networking, peer instruction, and whole-class discussions in which the teacher serves as a moderator.

Such instructional methods will require **the teacher's role to shift from dispensing information to facilitating learning.** New topics, whenever possible, should be introduced through real-life problem situations that encourage students to explore, formulate and test conjectures, prove generalizations, and discuss and apply the results of their investigations. As a result of such instruction, students should be able to learn mathematics both creatively and independently and thereby strengthen their confidence and skill in doing mathematics. **In fact, problem solving should not only be a means of instruction but also a goal.** The relationship of problem solving to other teaching strategies is very fundamental. One way students can obtain practice in using a problem solving process is for the learning situation to be one where they can discover for themselves the mathematics they are to learn. *Instructional Approaches: A Framework for Professional Practice* (1991) provides additional information to understand and implement a variety of approaches to teaching.

The use of technology in instruction can facilitate the teaching and learning of mathematics. Computer software can be used for class demonstrations and independently by students to explore additional examples, to perform independent investigations, to generate and summarize data as part of a project, or to complete assignments. More information on teaching mathematics using a variety of instructional strategies can be found in *Curriculum Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989); or from the *Instructional Strategies Series* of booklets published by SIDRU and SPDU. (See References, page 97.)

Adaptive Dimension

The adaptation of instruction to meet learner needs is an expectation inherent in the **Goals of Education** and is an essential ingredient of any consideration of Instructional Approaches. The Adaptive Dimension is defined as:

> . . . the concept of making adjustments in approved educational programs to accommodate diversity in student learning needs. It includes those practices the teacher undertakes to make curriculum, instruction, and the learning environment meaningful and appropriate for each student. (*The Adaptive Dimension in Core Curriculum*, Saskatchewan Education 1992, page 1)

The continuum of curricular programs authorized by Saskatchewan Education - Regular, Transitional, and Alternative Programs - recognizes the need for variation in curriculum content and delivery mechanism. As indicated in the continuum, adaptation may be required within each program, and therefore within each course of study. **Teachers are empowered to adjust the curriculum topics and material in order to meet student needs**; as professionals they must ensure that the instructional approaches are also adapted. This implies that teachers have at their "fingertips" a broad, strong repertoire of instructional strategies, methods, and skills and that conscious planning takes place to adapt these approaches to meet student needs.

The cues that some students' needs are not being adequately met come from a variety of sources. They may come to the perceptive teacher as a result of monitoring for comprehension during a lesson. The cue may come from a unit test, or from a student need or background deficiency that has been recognized for several years. A student's demonstrated knowledge of, or interest in, a particular topic may indicate that enrichment is appropriate. The adaptation required may vary from presenting the same content through a slightly different instructional method, to enriching the material because of a known information background deficit or to establishing an individual or small group enrichment activity. The duration of the adaptation may range from five minutes of individual assistance, to provision of an alternative or enrichment program. The diagnosis of the need may be handled adequately by the classroom teacher, or may require the expertise of other support specialists such as the school's resource teacher or other system-based personnel.

The recognition of the need for adaptive instruction is dependent upon the professional judgment of the teacher. The decision to initiate the adaptation may occur through the placement of students in programs other than those defined as regular. The most frequent application of the Adaptive Dimension will occur as teachers in regular classroom settings adjust their use of instructional materials, methods and the environment. Further information can be found in *The Adaptive Dimension in Core Curriculum* (1992).

The flexibility inherent in the Mathematics curricula accommodates the Adaptive Dimension. Mathematics teachers will have to take advantage of and create inservice opportunities to adjust their repertoire of instructional strategies, methods, and skills.

Gender Equity

Saskatchewan Education is committed to providing quality education for all students in the K to 12 system. It is recognized that expectations, based primarily on gender, limit students' ability to develop to their fullest potential. Continued efforts are required so that equality of benefit or outcome may be achieved. It is the responsibility of schools to decrease sex-role expectations and attitudes in an effort to create an educational environment free of gender bias. This can be facilitated by increased understanding and use of gender balanced material and strategies, and through further efforts to analyze current practice. Both females and males need encouragement to explore non-traditional as well as traditional options.

In order to meet the goal of gender equity in the K to 12 system, Saskatchewan Education is committed to efforts to bring about the reduction of gender bias which restricts the participation and choices of students. It is important that Saskatchewan curricula and classrooms reflect the variety of roles and the wide range of behaviours and attitudes available to all members of our society. The new curriculum strives to provide gender balanced content, activities, and teaching strategies described in inclusionary language. These actions will assist teachers to create an environment free of bias and enable both males and females to share in all experiences and opportunities which develop their abilities and talents to the fullest.

Teachers need to believe that both females and males can perform well in mathematics at all grade levels. Teachers should also become aware of the attitudes displayed by their students and help them to view themselves as able to achieve in mathematics. It is important to show students the relevance of mathematics to their own lives, choosing examples which come from the experience of females and males. **From an early age, students need to be made aware that most careers will require mathematics.**

Teachers need to be aware of their own interactions with students ensuring that everyone takes an active part. Being aware of interactions between students which may reinforce limiting behaviour or attitudes, and taking opportunities to discuss them, will help students to acquire a broader understanding of their own abilities and potential. All of these actions will support and enhance gender equity in mathematics and move toward improved teaching practice.

Indian and Métis Curriculum Perspectives

The integration of Indian and Métis content and perspectives into the K-12 curriculum fulfils a central recommendation of *Directions, The Five Year Action Plan for Native Curriculum Development* (1984), and *The Indian and Métis Education Policy from Kindergarten to Grade XII* (1989). In general, the policy states:

> Saskatchewan Education recognizes that the Indian and Métis peoples of the province are historically unique peoples and occupy a unique and rightful place in society today. Saskatchewan Education recognizes that education programs must meet the needs of Indian and Métis peoples, and that changes to existing programs are also necessary to benefit all students.

In a pluralistic society, the inclusion of Indian and Métis perspectives benefits all students. Cultural representation in all aspects of the school environment helps provide children with a positive group identity. Appropriate resources foster meaningful experiences for Indian and Métis students and promote the development of positive attitudes in all students towards Indian and Métis peoples. Awareness of one's own culture and the cultures of others, develops positive self-concept and enhances learning.

Saskatchewan Indian and Métis students come from varied cultural backgrounds and geographic areas encompassing northern, rural, and urban environments. Teachers must be given support that enables them to create instructional plans relevant to meeting diverse needs. Varied social, cultural, and linguistic backgrounds of Indian and Métis students imply a range of strengths and learning opportunities for teachers to draw upon. Explicit guidance, however, is needed to assist teachers in meeting the challenge by enabling them to make appropriate choices in broad areas of curriculum support. Anti-bias curricula, cross-cultural education, first and second language acquisition, and standard and non-standard usage of language are becoming increasingly important to classroom instruction. Care must be taken to ensure teachers utilize a variety of teaching methods that build upon the knowledge, cultures, and learning styles students possess. All curricula, including mathematics require adaptations to the content, instructional practices, and learning environment that reflect the needs of the students.

The following four points summarize the Department's expectations for the appropriate inclusion of Indian and Métis content in curriculum and instruction.

- Curricula and materials will concentrate on positive images of Indian, Métis, and Inuit peoples.
- Curricula and materials will reinforce and complement the beliefs and values of Indian, Métis, and Inuit peoples.
- Curricula and materials will include historical and contemporary issues.
- Curricula and materials will reflect the legal, political, social, economic, and regional diversity of Indian, Métis, and Inuit peoples.

Assessment and Evaluation

Why Consider Assessment and Evaluation?

Much research in education around the world is currently focusing on assessment and evaluation. It has become clear, as more and more research findings accumulate, that a broader range of attributes need to be assessed and evaluated than has been considered in the past. A wide variety of ways of doing this are suggested. Assessment and evaluation are best addressed from the viewpoint of selecting what appears most valid in meeting prescribed needs.

In *Student Evaluation: A Teacher Handbook* (1991) the difference between the various forms of evaluation is explained. Student evaluation focuses on the collection and interpretation of data which would indicate student progress. This, in combination with teacher self-evaluation and program evaluation, provides a full evaluation.

Clarification of Terms

To enhance understanding of the evaluation process it is useful to distinguish between the terms "**assessment**" and "**evaluation**". These terms are often used interchangeably which causes some confusion over their meaning. **Assessment** is a preliminary phase in the evaluation process. In this phase, various strategies are used to gather information about student progress. **Evaluation** is the weighing of assessment information against some standard (such as a curriculum learning objective) in order to make a judgment. Evaluation may then lead to decision and action.

There are three main types of student evaluation: **formative, summative**, and **diagnostic evaluation**. Assessment strategies are used to gather information for each type of evaluation.

Formative evaluation is an ongoing classroom process that keeps students and educators informed of students' progress towards program learning objectives. The main purpose of formative evaluation is to improve instruction and student learning. It provides teachers with valuable information upon which instructional modifications can be made. This type of evaluation helps teachers understand the degree to which students are learning the course material and the extent to which their knowledge, understandings, skills, and attitudes are developing. Students are provided direction for future learning and are encouraged to take responsibility for their own progress.

Summative evaluation occurs most often at the end of a unit of study. Its primary purpose is to determine what has been learned over a period of time, to summarize student progress, and to report on progress relative to curriculum objectives to students, parents, and educators.

Seldom are evaluations strictly formative or strictly summative. For example, summative evaluation can be used formatively to assist teachers in making decisions about changes to instructional strategies or other aspects of students' learning programs. Similarly, formative evaluation may be used to assist teachers in making summative judgments about student progress. However, it is important that teachers make clear to students the purpose of assessments and whether they will later be used summatively.

Diagnostic evaluation usually occurs at the beginning of the school year or before a unit of instruction. Its main purposes are to identify students who lack prerequisite knowledge, understanding, or skills, so that remedial help can be arranged; to identify gifted learners to ensure they are being sufficiently challenged; and to identify student interests. Diagnostic evaluation provides information essential to teachers in designing appropriate programs for students.

It is typical to conduct all three types of evaluation during the course of the school year.

Phases of the Evaluation Process

Although **evaluation** is not strictly sequential, it can be viewed **as a cyclical process including four phases: preparation, assessment, evaluation, and reflection.** The evaluation process involves the teacher as a decision maker throughout all four phases.

- In the **preparation phase**, decisions are made which identify what is to be evaluated, the type of evaluation (formative, summative, or diagnostic) to be used, the criteria against which student learning outcomes will be judged, and the most appropriate assessment strategies with which to gather information on student progress. The teacher's decisions in this phase form the basis for the remaining phases.

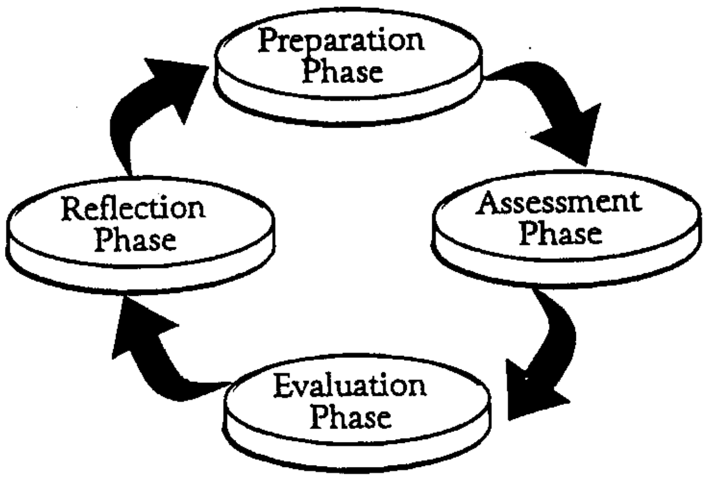
- During the **assessment phase**, the teacher identifies information-gathering strategies, collects student products, constructs or selects instruments, administers them to the student, and collects the information on student learning progress. The teacher continues to make decisions in this phase. The identification and elimination of bias (such as gender and culture bias) from the assessment strategies and instruments, and determining where, when, and how assessments will be conducted are examples of important considerations for the teacher in this phase of evaluation.

- During the **evaluation phase**, the teacher interprets the assessment information and makes judgments about student progress. Based on the judgments or evaluations, teachers make decisions about student learning programs and report on progress to students, parents, and appropriate school personnel.

- The **reflection phase** allows the teacher to consider the extent to which the previous phases in the evaluation process have been successful. Specifically, the teacher evaluates the utility and appropriateness of the assessment strategies used. Such reflection assists the teacher in making decisions concerning improvements or modifications to subsequent teaching and evaluation.

All four phases are included in formative, diagnostic, and summative evaluation processes. They are represented in Figure 1.

Figure 1. **Process of Student Evaluation**



Guiding Principles

Nine guiding principles are presented in the final report of the Minister's Advisory Committee on Evaluation and Monitoring, entitled, *Evaluation in Education* (1989). The purpose of these principles is to provide guidance on educational evaluation in several areas. One of these areas is student evaluation. The evaluation of student progress has a strong influence on both teaching and learning. If used appropriately, evaluation can promote learning, build confidence, and develop students' understanding of themselves.

Five general guiding principles provide a framework to assist teachers in planning for student evaluation:

1. Evaluation is an essential part of the teaching-learning process. It **should be a planned, continuous activity** which is closely linked to both curriculum and instruction.

2. Evaluation should be **guided by the intended learning outcomes of the curriculum**, and **a variety of assessment strategies should be used**.

3. Evaluation plans **should be communicated in advance**. Students should have opportunities for input to the evaluation process.

4. Evaluation should be **fair and equitable**. It should be sensitive to family, classroom, school, and community situations; it should be free of bias. Students should be given opportunities to demonstrate the extent of their knowledge, understandings, skills, and attitudes.

5. Evaluation **should help students.** It should provide positive feedback and encourage students to participate actively in their own learning.

Assessing Student Progress

Specific assessment strategies are selected or devised to gather information related to how well students are achieving the learning objectives of the curriculum. The assessment strategies used at any given time will depend on several factors such as the type of learning outcomes (knowledge, understanding, skill, attitude, value, or process), the subject area content, the instructional strategies used, the student's level of development, and the specific purpose of the evaluation.

Various assessment strategies are listed in Table 1 as a reference for teachers. The assessment strategies are not prescribed; rather, they are meant to serve only as suggestions, as the teacher must exercise professional judgment in determining which strategies suit the specific purpose of the evaluation. **It would be inappropriate for curriculum guides to give teachers specific formulas for assessing students.** Planning for assessment and evaluation must take into account unique circumstances and purposes which will vary. For further information on the various assessment strategies and types of instruments that can be used to collect and record information about student learning, refer to *Student Evaluation: A Teacher Handbook* (1991).

Common Essential Learnings (C.E.L.s) will be incorporated in the Foundational Objectives of each course. As each subject area is assessed and judgments are made, the C.E.L.s will form an integral part of the evaluation process within the area of study. For example, in a unit of instruction, some learning objectives will identify expected learning outcomes associated with C.E.L.s, but they will be embedded within the subject area content. Assessment strategies will be used to gather student progress information on C.E.L.s through the subject area. When all assessment information has been gathered, it will form the basis for an evaluation. **It is inappropriate to evaluate student progress in the Common Essential Learnings independent of the subject area content.**

A Reference List of Assessment Strategies

Table 1. **Assessment Strategies**

Student Classroom Performance

Anecdotal Records

Observation Checklist

Rating Scale

Contract

Laboratory Report

Portfolio

Test Station

Peer and Self-Assessment

Major Projects, Written Reports

Discussion Groups

Teacher Developed Test Formats

Matching Item

Multiple Choice

Oral

Short Answer

True/False

Essay

Performance Test

Student Assessment in Mathematics

At the beginning of any course, the Foundational Objectives and the learning objectives for the curriculum become the criteria to assess the student. These objectives may be attainable by the majority of students, but for some students, these objectives may not be attainable. Adaptations to instruction or procedures may be required.

A teacher must be aware that "graded" teaching resources and standardized tests are built on what is accepted as average for a student of a given age group and segment of society. In using standardized test, a teacher is assessing how a student matches these cultural standards over a very narrow range of skills. The results must be considered in that context. These standards may be unattainable by some students. Alternatively, some students may not reach full potential because they are not challenged but are allowed to remain at the acceptable "average". The Adaptive Dimension recognizes that the needs of all students must be considered for effective teaching and learning to occur.

The learning of mathematics is a cumulative process that occurs as experiences contribute to understanding. A numerical grade offers only a glimpse of a student's knowledge. If the goal of assessment is to obtain a valid and reliable picture of a student's understanding and achievement, evidence must come from a variety of sources. These sources may include oral presentations, written work, observations, or various combinations of these. Examples of written work include projects, homework assignments, journals, essays, quizzes, and exams. Records of a student's progress may include anecdotal records, portfolios, and mathematical journals. Rating scales and observation checklists are also helpful devices to record evidence of a student's continued growth in understanding. The advantage of using several kinds of assessments is that a student's understanding can be continuously monitored. In addition, because students differ in their perceptions and thinking styles, it is crucial that they are given the opportunity to demonstrate their individual capabilities. A single type of assessment can frustrate students, diminish their self-confidence, and make them feel anxious about mathematics. Examples of various templates for assessment and evaluation are included in this guide.

The assessment of a student's mathematical knowledge includes the ability to solve problems, to use the language of mathematics, to reason and analyze, to comprehend the key concepts and procedures, and to think and act in positive ways. Assessment should also examine the extent to which students have integrated and made sense of mathematical concepts and procedures and whether they can apply these concepts and procedures to situations that require creative and critical thinking.

Understanding concepts and their interrelationships is essential to interpreting a situation and deriving an appropriate plan of action. Knowing what procedures are appropriate and how to execute them is essential to carrying out the plan successfully.

Methods for assessing a student's ability to solve problems include observing the student solving problems individually, in small groups, or in class discussions. Other methods include listening to a student discuss problem-solving processes and analyzing tests, homework, journals, and essays. A rating scale is useful for assessing a student's problem solving skills. It may include rating a student's willingness to engage in problem solving, the use of a variety of strategies, facility in finding the solution to problems, and consistency in verifying the solution.

Assessment of a student's ability to communicate mathematically includes the meaning he/she attaches to the concepts and procedures of mathematics. It also involves his/her ability in talking about, writing about, understanding, and evaluating mathematical ideas. In assessing a student's ability to communicate, attention should be given to the clarity, precision, and appropriateness of mathematical terms and symbols. Discussion is also a splendid means of judging a student's ability to function as a critical participant in small groups or within the class.

The assessment of a student's ability to communicate through the use of computers and software, such as spreadsheets and data-base and function plotting programs, is also important. A student's ability to structure and present information with the use of technology can be assessed by determining if a student can use a spreadsheet or graph to simulate a situation or provide evidence for a conclusion.

An understanding of mathematical concepts involves more than mere recall of definitions and recognition of examples. It also encompasses a broad range of abilities. Assessment must include the aspects of conceptual understanding by focusing on a student's ability to discriminate between relevant and irrelevant attributes of a concept in selecting examples and non-examples, to represent concepts in various ways, and to recognize their various meanings. Observational checklists, anecdotal records or written reports may be used to assess such conceptual understanding.

Learning mathematics also includes developing a positive attitude towards mathematics. The assessment of a student's attitude requires information about her/his thinking and actions in a wide variety of situations. A student's attitudes are reflected in how he/she asks and answers questions, works on problems, and approaches new mathematics. Observations, homework assignments, journals, and oral presentations are all excellent ways to assess a student's mathematical attitude.

Program Evaluation

Program evaluation is a systematic process of gathering and analyzing information about some aspect of a school program in order to make a decision, or to communicate to others involved in the decision-making process. Program evaluation can be conducted in two ways: relatively informally at the classroom level, or more formally at the classroom, school, or school division levels.

At the classroom level, program evaluation is used to determine whether the program being presented to the students is meeting both their needs and the objectives prescribed by the province. Program evaluation is not necessarily conducted at the end of the program, but is an ongoing process. For example, if particular lessons appear to be poorly received by students, or if they do not seem to demonstrate the intended learning from a unit of study, the problem should be investigated and changes made. **By evaluating their programs at the classroom level, teachers become reflective practitioners.** The information gathered through program evaluation can assist teachers in program planning and in making decisions for improvement. Most program evaluations at the classroom level are relatively informal, but they should be done systematically. Such evaluations should include identification of the area of concern, collection and analysis of information, and judgment or decision making.

Formal program evaluation projects use a step-by-step problem-solving approach to identify the purpose of the evaluation, draft a proposal, collect and analyze information, and report the evaluation results. The initiative to conduct a formal program evaluation may originate from an individual teacher, a group of teachers, the principal, a staff committee, an entire staff, or central office. Evaluations are usually done by a team, so that a variety of skills are available and the work can be distributed. Formal program evaluations should be undertaken regularly to ensure programs are current.

To support formal school-based program evaluation activities, the Department has developed the *Saskatchewan School-Based Program Evaluation Resource Book* (1989) to be used in conjunction with an inservice package.

Curriculum Evaluation

During the decade of the 1990s, new curricula are being developed and implemented in Saskatchewan. Consequently, there is a need to know whether these new curricula are being effectively implemented and whether they are meeting the needs of students. Curriculum evaluation, at the provincial level, involves making judgments about the effectiveness of provincially authorized curricula.

Curriculum evaluation involves gathering information (the assessment phase) and making judgments or decisions based on the information collected (the evaluation phase), to determine how well the curriculum is performing. **The principal reason for curriculum evaluation is to plan improvements to the curriculum.** Such improvements might involve changes to the curriculum document and/or the provision of resources or inservice to teachers.

It is intended that curriculum evaluation be a shared, collaborative effort involving all of the major education partners in the province. Although the Department is responsible for conducting curriculum evaluations, various agencies and educational groups are involved. For instance, contractors are hired to design assessment instruments; teachers are involved in instrument development, validation, field testing, scoring, and data interpretation; and the cooperation of school divisions and school boards is necessary for the successful operation of the program.

In the assessment phase, information is gathered from students, teachers, and administrators. The information obtained from educators indicates the degree to which the curriculum is being implemented, the strengths and weaknesses of the curriculum, and the problems encountered in teaching it. The information from students indicates how well they are achieving the intended learning outcomes and provides indications about their attitudes toward the curriculum. Student information is gathered through the use of a variety of strategies including paper-and-pencil tests (objective and open-response), performance (hands-on) tests, interviews, surveys, and observation.

As part of the evaluation phase, assessment information is interpreted by representatives of all major education partners including the Department and classroom teachers. The information collected during the assessment phase is examined, and recommendations, generated by an interpretation panel, attempt to address areas in which improvements can be made. These recommendations are forwarded to the appropriate groups such as the Curriculum and Instruction Branch, school divisions and schools, universities, and educational organizations in the province.

All provincial curricula will be included within the scope of curriculum evaluation. Evaluations will be conducted during the implementation phase for new curricula, and regularly on a rotating basis thereafter. Curriculum evaluation is described in greater detail in *Curriculum Evaluation in Saskatchewan* (1990).

Teacher Self-Evaluation

There are two levels of teacher self-evaluation: reflection on day-to-day classroom instruction and professional self-evaluation.

Teachers refine their skills through reflecting upon elements of their instruction which include evaluation. The following questions may assist teachers in reflecting on their evaluations of student progress:

- Was there sufficient probing of student knowledge, understanding, skills, attitudes, and processes?
- Were the assessment strategies appropriate for the student information required?
- Were the assessment conditions conducive to the best possible student performance?
- Were the assessment strategies fair/appropriate for the levels of student abilities?
- Was the range of information collected from students sufficient to make interpretations and evaluate progress?
- Were the results of the evaluation meaningfully reported to students, parents, and other educators as appropriate?

Through reflection on questions like those above, teachers are able to improve their strategies for student evaluation.

It is also important for teachers, as professionals, to engage in self-evaluation. Teachers should take stock of their professional capabilities, set improvement targets, and participate in professional development activities. Some ways teachers can address their professional growth are by: reflecting on their own teaching; reading professional documents (e.g., articles, journals and books); attending workshops, professional conferences, and courses; and developing networks with other professionals in their fields.

Information Gathering and Record-keeping

Having summarized the various types of assessment and evaluation, it is obvious that large amounts of data are gathered by teachers, schools and school divisions, and the Department. It is **important that teachers maintain appropriate records** to ensure data are organized and accessible for making judgments and decisions. Records can be kept in a variety of ways; however, it is **recommended that teachers keep separate files on student progress** (student portfolios), **teachers' self-evaluations** (professional files), and **program evaluation**. Schools and school divisions also keep records of student enrollment and progress which support decision making at the local level. Saskatchewan Education is developing and implementing a comprehensive student record system with the capacity to register students K-12. This database will assist schools, school divisions, and the province to make informed decisions related to areas such as student mobility, dropout rates, retention rates, and student ethnicity. Saskatchewan Education also requires comprehensive information to make informed decisions at the provincial level in areas such as program and curriculum evaluation.

Conclusion

Evaluation is the reflective link between what ought to be and what is; therefore, it is an essential part of the educational process. The main purposes for evaluating are to facilitate student learning and to improve instruction. By continuously evaluating student progress, school programs, curriculum, and the effectiveness of instruction and evaluation, these purposes will be realized.

Program Organization

Conceptual Teaching

Concepts are the basis of formal education. Since formal education proceeds mainly through language and is highly concentrated, concepts are of great importance. Principles, generalizations, and rules of procedures are also formulated by means of concepts. Attitudes too, are grounded on concepts. Consequently, it is through the understanding of the fundamental mathematical concepts and their meaning or interpretation that students will make sense of mathematics.

One way an individual can respond to collections of objects is by distinguishing among them. Another way, even more important as a human capability, is by putting things into a class and responding to the class as a whole. The latter type of learning, which makes it possible for the individual to respond to things or events as a class, is called Concept Learning.

Concept teaching lends itself well to mathematics because it is process oriented. Concept teaching approaches are aimed at teaching students to think, to question and to discover rather than to memorize. A conceptual approach also encourages inductive thinking as students move from particular facts to generalizations. It enables students to sort more effectively through the multitude of mathematical information by stressing the commonality in the information. Students are provided with the opportunity to examine and experience and to develop a basis for enhancing their understanding of the concepts they form and acquire. This approach to learning encourages the development of Critical and Creative Thinking and promotes Independent Learning.

There are various strategies that are effective in teaching concepts. However, the intellectual aptitude of the student, what he/she may already know about the concept, the nature of the concept, its significance in the structure of other mathematical concepts, and the level of performance expected are all factors that a teacher should consider in choosing the most appropriate strategy. As a result, the student will develop the abilities to discriminate between relevant and irrelevant attributes of a concept in selecting examples and non-examples, to represent concepts in various ways, and to recognize their meanings.

Students with Learning Disabilities

Recent research has indicated that students with learning disabilities can learn higher concepts in mathematics through a strategy referred to as the Cognitive Assault Strategy (Miles and Forcht, 1995).

This strategy entails the use of a mentor who helps the students progress through a four-stage model. The four stages are: 1) demonstrating and modelling a problem-solving strategy, 2) helping with mathematics vocabulary, 3) assisting students in developing and verbalizing their strategy, and 4) teaching the students to write out their own procedural model for future reference.

Integration

Course content should be presented within the context of its application in daily living, integrated within the various branches of mathematics, and related to other academic disciplines. Integration may include any one of several forms.

- Interdisciplinary Studies - combining subjects
- Thematic - by topic or by concepts
- Holistic Approach - giving the big picture first
- Infusion - integrating technologies or teaching strategies, into the school program
- Integrated Brain work - allows for seeking and creating meaningful organization by the individual
- Using All Mind/Brain Functions in Learning - cognitive, affective, physical/sensing, and intuitive

Teachers need to be familiar with the mathematical competence required of students in the particular course of study, everyday life, and other academic disciplines. Cooperative planning and conferencing with other teachers is central to understanding differing contexts in which basic mathematical skills are used, and will assist teachers in providing practical learning experiences that encourage transfer of knowledge and skill.

Problem Solving

Problem solving plays an integral role in the mathematics program so that students are provided with some of the thinking and problem solving skills necessary to help explain the world around them. Problem solving is the process of accepting a challenge and striving to resolve it. It allows students to become skillful in selecting and identifying relevant conditions and concepts, searching for appropriate generalizations, formulating plans, and employing acquired skills.

In 1945, George **Polya** published the book *How To Solve It* in which he outlined a **four-step model** which could be used during the solution of problems. It involves **understanding the problem, devising a plan, carrying out the plan, and looking back.** Problem solvers can learn individual skills and strategies to use within this framework. As they broaden their knowledge, their facility for solving problems will improve. Polya's framework is not fixed. Although problem solvers may approach a solution in the order outlined, they will often return to earlier stages because they encounter an obstacle and it becomes obvious that another approach will work better.

Problem solving is a process which is learned by doing. Students will become better problem solvers if they think that the activity is important and relevant. This importance is enhanced by observing their teacher(s) solving problems and by their teacher(s) expecting them to do the same. Teaching the four-step problem solving model lends itself to incorporating the Common Essential Learnings; in particular, Critical and Creative Thinking.

When mathematical concepts and operations are introduced, they should often follow rather than precede problem-solving opportunities. Before the formal terms and symbols are presented, students should learn to approach problems in a variety of ways. The problems should consist of a good mix of process, realistic, and translation problems.

Process problems are those that generally can not be solved using routine procedures; rather, their solution typically involves the application of some problem solving strategy or heuristic. **Realistic problems** are not well defined and require some further specification and refinement, often have multiple solutions, require the collection of information, involve collaboration with other people, cannot be solved in a few minutes, and involve some personal commitment on the part of the student. **Translation** problems involve translating written or verbal statements into mathematical expressions and then performing an algorithm.

Some of the problem solving strategies that students should develop include:

- representing the problem situation with an appropriate diagram, model, or simulation;
- incorporating the data into an organized list, table or chart, suitable graph, or tree diagram;
- guessing and checking, substituting a guessed number, carrying out a calculation, checking the suitability of the answer, making an adjustment and repeating the process;
- changing a point of view;
- reconstructing and solving a related, but more simple problem;
- looking for a numerical or geometrical pattern which can be generalized;
- using logical thinking to eliminate possibilities; and,
- working backwards.

Estimation

Estimation is an important mathematical and life skill necessary for effective problem solving, calculations, and calculator use. The desired accuracy of an estimation depends entirely on its use. Whether to produce a rough estimate, fine estimate, or an exact answer depends on the ultimate use of the estimate. Most often practical situations involve estimations rather than exact numbers. It should also be remembered that estimation produces answers that are not exact, but that are adequate for making necessary decisions.

Mental Calculations

Mental calculation is an important and frequently used practical life skill. Despite the increasing advances in calculator technology, mental calculations will remain a convenient calculation tool for judging a narrow range of quantitative problems. It can also improve the efficiency of pencil and paper calculations by reducing the number of steps needed to work out a written calculation. The close relationship between estimating and calculating mentally implies that teachers cannot ignore instruction on mental calculations. After all, mental calculation is the cornerstone for all estimation procedures.

To assist students in developing the ability to calculate mentally it is beneficial to:

- encourage students to calculate mentally whenever possible;
- expect students to explain their methods whenever possible;
- practise mental calculations in a meaningful setting;
- treat mental calculations as a skill to be used in all areas of the program;
- not discourage capable students from skipping steps when performing written calculations;
- provide regular practice and review in learning the basic number facts;
- allow students to develop their own informal approaches to calculate and solve problems; and,
- help students recognize that there is no best method of calculating mentally.

Calculators in School Mathematics

Improvements in technology and its increased availability in schools have changed the focus of mathematics education. Calculators have become relatively inexpensive and plentiful in our society. Students encounter calculators in their everyday lives and should learn **how** to use them efficiently and **when** it is appropriate to use them. The use of calculators in school mathematics, from elementary through secondary levels, alters the content we teach and the concepts we emphasize. Less emphasis is placed on facility with paper-and-pencil calculations and more emphasis on mathematical concepts and their relationships.

Many people fear that the use of calculators in the mathematics classroom will discourage the learning of basic facts and the development of mental skills. Researchers indicate that students who regularly use calculators show either an achievement edge or no significant difference over those who do not (Owens, p. 232). Even when the testing occurs without the use of a calculator, those who learned the concepts with a calculator do just as well as those who did not have access to one. When calculators are used in testing procedures, students achieve much higher in problem solving as well as in basic operations. Students at all grades and ability levels benefit from the use of calculators. They have a greater motivation to work together, more self-confidence in problem solving, and more positive attitudes and enthusiasm about mathematics. They show more persistence and a willingness to seek alternative solutions. Researchers also indicate that the learning of basic facts and skills is enhanced through the use of calculators (Hembree & Dessart, p. 25). They do not replace manipulatives but can be used with them in the development of a number of concepts. When students are encouraged to explore with calculators they often experience some mathematics topics before the time they would otherwise.

Calculator use increases the importance of the ability to estimate answers and to check the appropriateness of the results of calculations. Many students lack these abilities. They pay little attention to the reasonableness of their answers. Development of these abilities needs to be encouraged in students. Teachers need to develop in students the attitude that a calculator's answer needs to be checked for reasonableness. With the increased use of calculators, estimation and mental mathematics strategies become even more important.

As early as 1980, NCTM's *Agenda for Action* recommended that "mathematics take full advantage of the power of calculators and computers at all grade levels." (p. 1) However, in order to realize the potential of calculators, **students and teachers must develop the skills needed to apply the technology effectively**. The least beneficial activities with a calculator are as drill and practice and as a check on paper-and-pencil calculations. Rather, the calculator should be used as a tool to solve realistic problems, to discover mathematical patterns and relationships, to explore number properties and to develop estimation skills and number sense. Show students the calculator's limitations by giving a problem that is solved more easily using mental calculation.

In elementary grade levels, students can use calculators along with manipulatives to develop number concepts, counting skills and place value concepts. The constant function feature can be used to investigate the results of counting, to understand the concepts of multiples and powers, to develop an intuitive understanding of division as repeated subtraction, and to discover divisibility rules. At the middle level, the calculator can be used to provide the opportunity for all students, regardless of their knowledge of basic facts, to pursue other mathematical topics and to realize the usefulness, fun and power of mathematics. Students at this level are interested in topics like compound interest and population growth. More realistic and interesting problems can be solved if the calculator is used to do the computations. Frequently at this level, students have difficulty distinguishing between the concepts of perimeter, area and volume. Using a calculator, they can be assigned an increased number of problems to give them the experience necessary to master these concepts. At the secondary level, the graphing calculator allows new approaches for developing and reinforcing concepts such as those related to functions. At all grade levels, students can use a calculator to generate, in a short time, several examples of patterns, to organize the examples according to patterns formed, and to make generalizations and predictions from the examples. In general, calculators should be used when doing problem solving but should be removed when knowledge of operations on numbers is required.

If we are planning to integrate calculator use throughout our mathematics courses, we must inform parents of our intentions and provide them with answers to their concerns. Parents can be informed through a variety of ways including newsletters, workshops, open houses, and activities sent home to be done by parent and student. When parents see the type of activities students are doing, they are more likely to be supportive of the program. Assure parents that students are still expected to master the basic facts and computational skills. Become knowledgeable about the research on calculators, recognize the benefits of calculator use and develop a clear calculator policy that is understood by every student, parent, and teacher.

Some parents are concerned that students will not think if they use calculators and will become too dependent on them. The student must still know what keys to press and what operations to use. Students must still understand the mathematics in order to choose the appropriate keys and operations. Calculators actually free students from tedious computations and allow them to concentrate on the thinking processes required to solve a problem. If students understand concepts and become more proficient in estimation and mental arithmetic, they will not be limited to doing calculations with a calculator and will realize that it is only one tool. At times, estimation, mental mathematics or paper-and-pencil are more appropriate. The basic skills we require in today's world are not proficiency with operations but rather those of reasoning, communicating, problem solving and applying knowledge to new situations.

NCTM Standards identifies five goals of mathematics for students:

- Learning to value mathematics.
- Becoming confident in one's own ability.
- Becoming a mathematical problem solver.
- Learning to communicate mathematically.
- Learning to reason mathematically.

All of these are enhanced and promoted through the proper use of calculators. "Calculators and computers for users of mathematics, like word processors for writers, are tools that simplify, but do not accomplish, the work at hand." (National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*, 1989, p.8) If we are to prepare our students of today for the world of tomorrow, we must develop their ability to use a calculator effectively and efficiently.

An important implementation strategy is to obtain a classroom set of calculators. It is advantageous to have every student equipped with the same model of calculator so that instructions given apply to all. Failing this, every classroom should have some calculators to lend to those students who do not have their own. In these circumstances, both instructional and management problems multiply as instruction will need to be given for each of the different models students are using. It is very useful to have a calculator for the overhead projector to use for group instruction. Classroom sets of calculators can be managed in a variety of ways. Some schools have a wooden box built with rows and columns separated out for the return of each calculator. The rows and columns correspond to the student's location in the classroom and all calculators must be replaced before the students leave. Another suggestion is to place the calculators in a bag that just holds the number you have. Students do not leave the room until the bag is full.

Microcomputers

The microcomputer has many beneficial applications in mathematics. It has been identified as a tool for students to use to explore and discover concepts, by assisting with the transition from concrete representation to the more abstract mathematical stages of learning. Appropriate computer software can be the link between concrete manipulatives and the symbolic representation. Using a combination of manipulatives and corresponding computer software supports the constructivist perspective that learning is a process of constructing, building and fitting ideas, and making connections. By giving students the opportunity to experiment with various representations, we are offering more models with which to make these connections.

Computer hardware and software is becoming increasingly sophisticated in design in areas such as problem solving, manipulating, discovering, creative programming, games, tutorials, drill and practice, and managing. Interactive computer software holds great promise for application in the mathematics classroom. Its value in creating geometric displays, organizing and graphing data, simulating real-life situations, demonstrating mathematical relationships, and generating numerical sequences and patterns is evident. Word processing software, electronic spreadsheets, and database management software are all useful in Secondary Level mathematics. **Within these applications it is crucial that software correlates with the curriculum and the ability levels of the students.**

The microcomputer also has tremendous potential to help students with special needs. Students with learning difficulties, those who require enrichment activities, and students in multi-graded classrooms can all benefit from the individual assistance a computer can offer.

Manipulatives

Manipulative materials provide students with a concrete base upon which to build concepts and skills. **Students who experience difficulty with mathematical processes should** be given opportunities to develop an understanding of number operations and relationships through tactile and visual learning activities. Students also need to observe, question, verbalize, and discuss the relationships at the concrete level, and eventually to translate relationships into abstract symbols of mathematics. While the purpose of manipulatives is to help students understand and remember, each student should become efficient in making application of concepts in their abstract form. It is important that secondary teachers supplement and alternate the use of manipulative materials with other abstract learning strategies and activities.

Aids for Planning

- Scope and Sequence (by course)
- Scope and Sequence (by strand)

Scope and Sequence for Secondary Mathematics (by course)

Mathematics 10

Major Concepts	Number of Hours
A. Linear Equations and Inequalities	12
B. Relations, Linear Functions, and Variation Part I: Theory • Linear Functions • Linear Equations • Slope Part II: Applications • Direct Variation • Partial Variation • Applications - Arithmetic Sequences	28
C. Consumer Mathematics • Income • Budget	10
D. Lines and Line Segments • Parallel Lines • Perpendicular Lines	5
E. Angles and Polygons • n-gons • Triangles • Quadrilaterals • Parallelograms	10-15
F. Review of Algebraic Skills • Numbers and Operations • Exponents • Polynomials	5-10
* Optional Topics (objectives are not provided) • Direction Vectors a) on a Number Line b) in a Plane • Surveying • Integration with other subject areas	
Total	70-80

• The remaining available time may be spent on additional practice, enrichment, or extension of the course.

In this scope and sequence, each of the Foundational Objectives will be supported by the learning objectives numbered in each case. **The numbering in parentheses after each Foundational Objective is a coding that could be employed in any specific assessment.**

Objectives

A. Linear Equations and Inequalities

Foundational Objective

- To demonstrate the ability to solve linear equations and inequalities (10 01 01). Supported by the following learning objectives:

1. To solve and verify the following types of equations in one variable by applying formal algebraic rules: equations containing variables on both sides of the equal sign, equations containing parentheses, equations containing fraction or decimal coefficients.

2. To solve a formula for an indicated variable.

3. To solve, graph, and verify linear inequalities in one variable.

4. To translate English phrases into mathematical terms and vice versa.

5. To solve real-world problems using various problem-solving strategies.

B. Relations, Linear Functions, and Variation

Foundational Objectives

- To be aware that the graph of a first degree equation of two variables is a line, and conversely that the equation of a line is a first degree equation of two variables (10 02 01). Supported by learning objectives 1, 2, 3, 4, 5, and 9.

- To produce graphs of linear equations, and conversely, when given key information of the graph, find its equation (10 02 02). Supported by learning objectives 6, 7, 8, 10, 13, 14, 15, 16, 17, and 18.

- To use the knowledge of linear functions and equations to solve problems involving direct and partial variation (10 02 03). Supported by learning objectives 19 to 23.

- To use the concept of slope to measure the steepness of a line (10 02 04). Supported by learning objectives 11 and 12.

- To demonstrate the ability to work with arithmetic sequences (10 02 05). Supported by learning objectives 24 to 27.

Part I: Theory

1. To define the following terms: relation, ordered pair, abscissa, ordinate, function, linear function, slope, x-intercept, y-intercept, ratio, proportion, direct variation, partial variation.

2. To identify and express examples of relations in the real world.

3. To graph ordered pairs in the Cartesian coordinate plane.

4. To graph real-world relations in the Cartesian coordinate plane.

5. To read information from a graph.

6. To identify, graph, and interpret examples of linear functions describing real-world situations.

7. To graph a linear function using a table of values.

8. To determine if a relation is a function by employing the vertical line test.

9. To solve equations in two variables given the domain of one of the variables.

10. To determine if an ordered pair is a solution to the linear equation.

11. To calculate the slope of a line graphically and algebraically when given two points that lie on the line.

12. To determine the slope of horizontal lines, vertical lines, parallel lines, and perpendicular lines.

13. To graph a linear equation using the x- and y-intercepts.

14. To graph a linear equation using the slope and y-intercept.

15. To write the equation of a line when given the slope and one point on the line.

16. To write the equation of a line when given two points that lie on the line.

17. To construct scatterplots from real-world data.

18. To interpret and critically analyze these constructed scatterplots.

Part II: Applications

19. To identify, describe, and interpret examples of direct variation in real-world situations.

20. To solve proportions involving direct variation.

21. To solve problems involving direct variation.

22. To identify partial variation.

23. To solve problems involving partial variation.

24. To define, illustrate, and identify an arithmetic sequence.

25. To determine the n^{th} term of an arithmetic sequence.

26. To define arithmetic means, and to determine the required arithmetic means between given terms.

27. To calculate the sum of an arithmetic series.

C. Consumer Mathematics

Foundational Objectives

- To apply simple mathematics to assist in the calculation and estimation of income and expenses and to develop a budget to guide current and future planning (10 03 01). Supported by learning objectives 1 to 9.

- To communicate a summary of financial projections in appropriate reports, tables, and graphs (10 03 02). Supported by learning objectives 10 to 12.

1. To calculate weekly gross wages involving regular pay, overtime pay, and piecework earnings.

2. To calculate earnings for straight commission, or base wage plus commission.

3. To determine the difference between gross pay and net pay.

4. To calculate weekly, monthly, and yearly net pay.

5. To define and explain the purpose of a budget.

6. To determine and calculate monthly fixed expenditures.

7. To investigate the guidelines in developing a budget.

8. To plan a budget based on percentages allotted to various categories as suggested by financial institutions.

9. To calculate the portion of total income spent on each category using percents.

10. To draw graphs (including circle graphs) of budget figures, using appropriate software.

11. To calculate the actual amount of money to be spent on each category using the predetermined percentages.

12. To adjust a budget to changes in expenses.

D. Lines and Line Segments

Foundational Objective

- To develop an informal understanding of the relationships between lines (10 04 01). Supported by the following learning objectives.

1. To define line segment, ray, line, bisector, perpendicular line, perpendicular bisector, transversal, alternate interior angles, corresponding angles, same-side interior angles.

2. To identify and calculate the measures of the following angles formed by parallel lines: corresponding angles, alternate interior angles, and same-side interior angles.

3. To solve word problems involving angles formed by parallel lines.

4. To construct informally a line parallel to a given line through a point not on the line.

5. To construct informally a line perpendicular to a given line from a point on the line.

6. To construct informally a line perpendicular to a given line through a point not on the line.

7. To construct informally the perpendicular bisector of a line segment.

Construct informally means employing the mira or paperfolding. Traditional straight edge and compass constructions may be used as enrichment.

E. Angles and Polygons

Foundational Objectives

- To identify and apply common properties of triangles, special quadrilaterals, and n-gons (10 05 01). Supported by learning objectives 1, 2, 3, 4, 5, 8, 9, and 10.
- To apply the special quadrilaterals to real-world situations (10 05 02). Supported by learning objectives 6 and 7.
- To develop an understanding of Pythagoras' Theorem, the primary trigonometric ratios, and their applications (10 05 03). Supported by learning objectives 11 to 17.

1. To define and illustrate by drawing the following: acute angle, right angle, obtuse angle, straight angle, reflex angle, complementary angles, supplementary angles, adjacent angles, vertically opposite angles, congruent angles, central angles of a regular polygon.

2. To solve word problems involving the angles stated in #1.

3. To define and illustrate the following polygons: convex, non-convex, regular, quadrilateral parallelogram, rectangle, rhombus, square, trapezoid, and isosceles trapezoid.

4. To classify quadrilaterals as trapezoids, isosceles trapezoids, parallelograms, rectangles, rhombuses, and squares.

5. To construct informally parallelograms, rectangles, rhombuses, and squares.

6. To state and apply the properties of parallelograms.

7. To determine the sum of the measures of the interior and exterior angles of a convex polygon of n sides.

8. To determine the measure of a central angle in a regular n-gon.

9. To determine the measures of the interior and exterior angles of regular n-gons.

10. To determine the number of diagonals in a polygon of n sides.

11. To calculate to two decimal places the length of a missing side of a right triangle using the Pythagorean Theorem.

12. To solve word problems using the Pythagorean Theorem.

13. To determine if a triangle is a right triangle by using the converse of the Pythagorean Theorem.

14. To determine the value of the three primary trigonometric ratios by using a calculator.

15. To determine the measure of an angle given the value of one trigonometric ratio of the angle by using a calculator.

16. To calculate the measure of an angle or the length of a side of a right triangle using the tangent, sine, and cosine ratios.

17. To solve word problems that involve trigonometric ratios using a calculator.

F. Review of Algebraic Skills

This is a review of grade 9 and is intended to precede Mathematics 20. Some of these objectives have not been expanded in this guide.

Foundational Objective

- To make the transition from arithmetic skills to algebraic skills (10 06 01). Supported by the following learning objectives.

1. Numbers and Operations
   a) To represent numbers on a number line.
   b) To convert fractions to terminating or repeating decimals, and vice versa.
   c) To add and subtract rational numbers.
   d) To multiply and divide rational numbers.
   e) To use the order of operations in evaluating rational arithmetic expressions.

2. Exponents
   a) To evaluate a positive power of a numerical base.
   b) To evaluate multiplication and division of positive powers of the same numerical base.
   c) To perform the product and quotient exponent properties with variable bases.
   d) To write numbers in scientific notation and vice versa.
   e) To perform multiplication and division of numbers expressed in scientific notation.

3. Polynomials
   a) To add and subtract polynomials.
   b) To multiply a monomial by a monomial.
   c) To multiply a polynomial by a monomial.
   d) To multiply a binomial by a binomial.
   e) To divide a monomial by a monomial
   f) To divide a polynomial by a monomial.

Mathematics 20

Major Concepts	Number of Hours
Mathematics 20	8
A. Irrational Numbers • Square root operations	
B. Consumer Mathematics • Credit and Loans • Income and Property Taxes	8
C. Polynomials and Rational Expressions • Factoring - up to trinomials ($ax^2 + bx + c$) • Laws of exponents - integral exponents • Operations - (+, -, x, /)	15
D. Quadratic Functions • Graphing various types of quadratic functions • Determine properties of each constant for the types used in section a) • Analyze and solve real-world problems based on a) and b)	10
E. Quadratic Equations • Solve by factoring • Solve by 'square root property' • Solve radical equations - one radicand • Solve real-world problems based on a), b), c)	10
F. Probability • Experimental • Theoretical	4
G. Angles and Polygons • Congruent triangles - informal, guided proofs • Similar polygons • Real-world problems • Areas and volumes of similar figures	15-20
H. Circles • Some relationships of tangents, chords, arcs • Real-world problems	5
* Optional Topics (objectives are not provided) • Consumer Price Index • Insurance	
Total	75-80

- The remaining available time may be spent on additional practice, enrichment, or extension of the course.

Objectives

A. Irrational Numbers

Foundational Objective

- To identify an irrational number and to demonstrate the ability to add, subtract, multiply, and divide square root radicals (10 01 01). Supported by the following learning objectives.

1. To define and illustrate, by means of examples, the term absolute value.
2. To express square root radicals as mixed radicals in simplest form.
3. To add, subtract, multiply, and divide square root radicals.
4. To rationalize monomial denominators.

B. Consumer Mathematics

Foundational Objectives

- To demonstrate an understanding of credit and to employ the appropriate mathematics in determining the cost to the consumer of various types of credit (10 02 01). Supported by learning objectives 1 to 10.
- To display an awareness of the kinds of taxes encountered by the consumer, and to demonstrate the ability to calculate these taxes using the appropriate mathematics. (10 02 02). Supported by learning objectives 11 to 14.

1. To define credit and determine its appropriate use.
2. To compare the advantages and disadvantages of various credit cards.
3. To calculate the monthly interest charges and service charges on an unpaid credit card balance.
4. To identify and compare an instalment charge account and a thirty-day account.
5. To identify the characteristics of a personal loan.
6. To compare the cost of a consumer loan from various institutions.
7. To calculate the monthly payments for a loan, using formulas, tables, calculators, and computers.
8. a) To determine the total cost of a personal loan.
   b) To determine the percent of the total amount repaid (or borrowed) which is devoted to interest.
9. To determine the importance of a credit rating.
10. To describe how to establish a good credit rating.
11. To calculate mill rates and property taxes.
12. To calculate discounts or penalties on taxes due, depending on when they are paid.
13. To determine and calculate permissible deductions from total income.
14. To calculate the tax payable on taxable income.

C. Polynomials and Rational Expressions

Foundational Objectives

- To demonstrate the ability to factor polynomial expressions, including trinomials of the type $ax^2 + bx + c$ (10 03 01). Supported by learning objectives 1 and 6.
- To demonstrate the ability to correctly simplify expressions that contain positive and negative integral exponents (10 03 02). Supported by learning objectives 3 and 4.

- To demonstrate the ability to add, subtract, multiply, and divide rational expressions with monomial denominators (10 03 03). Supported by learning objectives 2, 5, 7, and 8.

  1. To factor polynomials of the following types: common factor, grouping, difference of squares, trinomial squares, trinomials where a=1 and a≠1, and combinations of all preceding types.
  2. To divide a polynomial by a binomial, by factoring, and by long division.
  3. To evaluate powers with positive and negative exponents.
  4. To simplify variable expressions with integral exponents using the following properties of exponents: product, quotient, power of a product, power of a quotient, negative exponent, and zero exponent.
  5. To determine the non-permissible values for the variable in rational expressions.
  6. To simplify rational expressions by factoring.
  7. To multiply and divide rational expressions.
  8. To add and subtract rational expressions involving like and unlike **monomial** denominators.

D. Quadratic Functions

Foundational Objectives

- To be aware that the graph of an equation in two variables, where only one variable is of degree two, is a parabola (10 04 01). Supported by learning objective 1.
- To draw graphs of equations representing parabolas (10 04 02). Supported by learning objectives 2 and 3.
- To demonstrate an ability to interpret an equation of the form $y=a(x-p)^2=q$, as to the effect the values of a, p, and q have on the graph (10 04 03). Supported by learning objectives 4 and 5.

  1. To define a quadratic function.
  2. To identify, graph, and determine the properties of quadratic functions of the following forms: $f(x)=ax^2$, $f(x)=x^2+q$, $f(x)=(x-p)^2$, $f(x)=a(x-p)^2+q$.
  3. To determine the domain and range from the graph of a quadratic function.
  4. To analyze the graphs of quadratic functions that depict real-world situations.
  5. To solve problems involving the graphs of quadratic functions that depict real-world situations.

E. Quadratic Equations

Foundational Objectives

- To demonstrate the ability to solve quadratic equations by factoring, and by taking the square root of both sides of an equation (10 05 01). Supported by learning objectives 1, 2, and 3.
- To demonstrate the ability to solve equations containing one radical (10 05 02). Supported by learning objectives 4 and 5.

  1. To solve quadratic equations by a) factoring, and b) by taking the square roots of both sides of an equation.
  2. To calculate the exact value of the length of a side of a right triangle using the Pythagorean Theorem.
  3. To solve word problems involving quadratic equations.
  4. To solve and verify radical equations containing one radicand.

5. To solve problems that involve equations which contain radicals.

F. Probability

Foundational Objective

- To appreciate the role of probability in understanding everyday situations (10 06 01). Supported by the following learning objectives.

1. To list the sample space and events for a random experiment.
2. To calculate the experimental probability of simple events by performing repeated experiments.
3. To calculate the theoretical probability of an event, and the probability of its complement.

G. Angles and Polygons

Foundational Objectives

- To develop the ability to identify pairs of congruent triangles and to employ the congruence postulates SSS, SAS, ASA, AAS, or HL in guided proofs showing such congruences (10 07 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to apply the concepts of similar polygons and scale factors to determine the surface area and/or volume of similar polygons or solids (10 07 02). Supported by learning objectives 8 to 15.
- To provide a reasonable explanation for congruences of pairs of triangles, or for corresponding parts of congruent triangles (10 07 03). Supported by learning objectives 6 and 7.

1. To construct informally and formally congruent angles, and congruent triangles.
2. To determine the properties of congruent triangles.
3. To identify and state corresponding parts of congruent triangles.
4. To determine whether triangles are congruent by SSS, SAS, ASA, AAS, or HL.
5. To prove that two triangles are congruent by supplying the statements and reasons in a guided deductive proof.
6. To prove triangles congruent by SSS, SAS, AAS, ASA, or HL in a two-column deductive proof or paragraph form.
7. To prove corresponding parts of congruent triangles are congruent.
8. To identify similar polygons.
9. To determine the measure of corresponding angles in two similar polygons.
10. To calculate the scale factor of two similar polygons.
11. To calculate the length of a missing side of two similar polygons.
12. To show that two triangles are similar by the Angle Angle Similarity Theorem (Postulate in some resource texts).
13. To calculate the length of a missing side in two similar right triangles.
14. To solve problems involving similar triangles, and other polygons.
15. To determine surface area and volumes of similar polygons or solids.

H. Circles

Foundational Objective

- To develop and display an understanding of certain relationships of the chords, tangents, and arcs of a circle (10 08 01). Supported by the following learning objectives.

Objectives

A. Permutations and Combinations

Foundational Objectives

- To demonstrate the ability to determine the number of permutations in a given situation (10 01 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to determine the number of combinations in a given situation. (10 01 02). Supported by learning objectives 6 and 7.

1. To apply the fundamental counting principles to determine the number of possibilities that exist in a given situation.
2. To determine the number of permutations of n objects. $({}_nP_n = n!)$
3. To determine the number of permutations of n different objects, taken r at a time. $({}_nP_r)$
4. To determine the number of permutations of n objects, not all different.
5. To determine the number of permutations of n objects arranged in a circle.
6. To determine the number of combinations of n objects, taken r at a time.
7. To determine the number of combinations formed from more than one subset.

B. Data Analysis

Foundational Objectives

- To demonstrate developed skills and understanding in collecting and displaying a set of data for a given situation. (10 02 01). Supported by learning objectives 1 to 4.
- To provide reasonable explanations of the interpretation of a set of data. (10 02 02). Supported by learning objectives 5 and 6.

1. To list, and describe, the methods used to collect data.
2. To obtain data for real-world situations by using simulations (such as Monte Carlo simulations).
3. To review the methods for determine the measures of central tendency.
4. To construct box and whisker plots from simulated data.
5. To define and utilize the concept of percentiles (including the first, second and third quartiles).
6. To solve related problems using statistical inference.

C. Polynomials and Rational Expressions

Foundational Objectives

- To demonstrate ability in the addition, subtraction, multiplication, and division of rational expressions. (10 03 01). Supported by learning objectives 1 to 7.
- To demonstrate ability in solving equations involving rational expressions. (10 03 02). Supported by learning objectives 8 and 9.

1. To factor the difference of squares of special polynomials.
2. To factor the sum and difference of cubes.
3. To factor polynomials using the factor theorem.
4. To use the remainder theorem to determine the remainder when a polynomial is divided by (x-r).
5. To simplify rational expressions involving opposites.

6. To add and subtract rational expressions with polynomial denominators

7. To multiply and divide rational expressions involving opposites.

8. To solve and verify linear equations in one variable involving rational algebraic expressions (including polynomial denominators).

9. To solve and verify the solutions of quadratic equations involving rational algebraic expressions.

D. Exponents and Radicals

Foundational Objectives

- To be able to illustrate the relationship between the radical and exponential forms of an equation. (10 04 01). Supported by learning objectives 1, 3, and 5.

- To demonstrate the ability to work with operations involving radical numbers. (10 04 02). Supported by learning objectives 2, 4, 6, and 7.

- To be able to solve equations involving radicals, and to be able to justify the solutions. (10 04 03). Supported by learning objects 8 to 11.

1. To evaluate powers with rational exponents.

2. To apply the laws of exponents to simplify expressions involving rational exponents.

3. To write exponential expressions in radical form.

4. To simplify square root and cube root expressions.

5. To write radical expressions in exponential form.

6. To add, subtract, multiply and divide square root and cube root expressions.

7. To rationalize monomial and binomial denominators in radical expressions.

8. To solve and verify the solutions of quadratic equations by factoring, completing the trinomial square and using the quadratic formula.

9. To solve and verify equations involving absolute value.

10. To solve radical equations with two unlike radicands.

11. To solve word problems involving radical equations.

E. Relations and Functions

Foundational Objectives

- To demonstrate the ability to work with functional notation and related operations. (10 05 01). Supported by learning objectives 1, 2, and 3.

- To be able to produce graphs of relations and functions and to be able to denote which graphs represent functions. (10 05 02). Supported by learning objectives 4 and 8.

- To demonstrate the ability to interpret graphs representing functions and to identify key points of these graphs. (10 05 03). Supported by learning objects 5, 6, and 7.

1. To evaluate functions using functional notation.

2. To perform indicated operations on functions using function notation.

3. To form the composite of two or more given functions.

4. To determine if a function is one-to-one or many-to-one.

5. To write the equation of a line in standard form using: two intercepts, slope and one intercept, one point and the equation of a parallel line, and one point and the equation of a perpendicular line.

6. To solve real-world problems involving quadratic functions by analyzing their graphs.

7. To graph quadratic functions of the standard form $f(x) = a(x-p)^2+q$ by determining the vertex, axis of symmetry, concavity, maximum or minimum values, domain, range, and zeros.

8. To graph quadratic functions of the general form $f(x) = ax^2+bx+c$, by completing the trinomial square and converting to one of the standard forms.

9. To identify and graph examples of inverse variation taken from real-world situations.

10. To state the domain and range, along with any restrictions, for the graphs of inverse variations.

11. To determine the constant of proportionality of an inverse relation.

12. To solve problems that involve inverse variation.

F. Systems of Linear Equations

Foundational Objectives

- To be able to identify the number of possible solutions of a system of linear equations. (10 06 01). Supported by learning objective 3.

- To demonstrate the ability to solve a system of linear equations. (10 06 02). Supported by learning objectives 1, 2, and 4.

1. To solve and verify systems of linear equations in two unknowns by the following methods: graphic, substitution, and elimination.

2. To solve linear systems in two unknowns that have rational coefficients and verify the solutions.

3. To recognize the characteristics of linear equations in two variables with graphs that are inconsistent, consistent-dependent, or consistent-independent.

4. To solve word problems involving linear systems in two variables.

G. Angles and Polygons

Foundational Objectives

- To demonstrate the ability to determine trigonometric ratios in a given situation, and apply these ratios to solving real-world problems. (10 07 01). Supported by the following learning objectives.

1. To sketch an angle in standard position.

2. To determine the distance from the origin to a point on the terminal arm of an angle.

3. To calculate the distance between two ordered pairs in the coordinate plane.

4. To determine the coordinates of the midpoint of a segment.

5. To determine the value of the six trigonometric ratios when given a point on the terminal arm of an angle in standard position. (x,y,r).

6. To determine coterminal angles for a given angle.

7. To determine the reference angle for positive or negative angles.

8. To determine the values for the six trigonometric ratios when given one trigonometric ratio and the quadrant in which the angle terminates.

9. To determine the value for the trigonometric ratios by using a calculator.

10. To apply the trigonometric ratios to problems involving right triangles.

11. To determine the relationships among the sides of each special right triangle (45°-45°-90° and 30°-60°-90°).

12. To calculate the length of the missing sides of the special right triangles when given the exact value of one side.

7. To graph quadratic functions of the standard form $f(x)=a(x-p)^2+q$, by determining the vertex, axis of symmetry, concavity, maximum or minimum values, domain range, and zeroes.

8. To graph quadratic functions of the general form $f(x)=ax^2+bx+c$, by completing the trinomial square and converting to one of the standard forms.

9. To identify and graph examples of inverse variation taken from real-world situations.

10. To state the domain and range, along with any restrictions, for the graphs of inverse variations.

11. To determine the constant of proportionality of an inverse relation.

12. To solve problems that involve inverse variation.

F. Systems of Linear Equations

Foundational Objectives

- To be able to identify the number of possible solutions of a system of linear equations (10 06 01). Supported by learning objective 3.

- To demonstrate the ability to solve a system of linear equations (10 06 02). Supported by learning objectives 1, 2, and 4.

1. To solve and verify systems of linear equations in two unknowns by the following methods: graphic, substitution, and elimination.

2. To solve linear systems in two unknowns that have rational coefficients and verify the solutions.

3. To recognize the characteristics of linear equations in two variables with graphs that are inconsistent, consistent-dependent, or consistent-independent.

4. To solve word problems involving linear systems in two variables.

G. Angles and Polygons

Foundational Objectives

- To demonstrate the ability to determine trigonometric ratios in a given situation, and apply these ratios to solving real-world problems (10 07 01). Supported by the following learning objectives.

1. To sketch an angle in standard position.

2. To determine the distance from the origin to a point on the terminal arm of an angle in standard position.

3. To calculate the distance between two ordered pairs in the coordinate plane.

4. To determine the coordinates of the midpoint of a segment.

5. To determine the value of the six trigonometric ratios when given a point on the terminal arm of an angle in standard position (x, y, r).

6. To determine coterminal angles for a given angle.

7. To determine the reference angle for positive or negative angles.

8. To determine the values for the six trigonometric ratios, when given one trigonometric ratio and the quadrant in which the angle terminates.

9. To determine the values for the trigonometric ratios by using a calculator.

10. To apply the trigonometric ratios to problems involving right triangles.

11. To determine the relationships among the sides of each special right triangle (45-45-90 and 30-60-90).

12. To calculate the length of the missing sides of the special right triangles when given the exact value of one side.

Mathematics B 30

Major Concepts	Number of Hours
Mathematics B 30	
A. Probability • Independent, dependent, mutually exclusive events • Binomial Theorem	12
B. Data Analysis • Data distributions - normal, skewed • Standard deviation - calculation, interpretation • z-scores - calculation and usage • Problem solving	10
C. Matrices • Operations with matrices • Row operations • Solving equations • Linear Programming - Systems of Inequalities	15
D. Complex Numbers • Operations with complex numbers	5
E. Quadratic Equations • Quadratic Formula • Nature of Roots • Equations of degree greater than two • Quadratic inequalities	10
F. Polynomial and Rational Functions • Sketch and analyze • Inverse of a function • Reciprocal of a function	10
G. Exponential and Logarithmic Functions • Laws of exponents • Graphs of these functions • Solving equations and problems • Geometric sequences and series • Problems	18
* Optional Topics (objectives are not provided) • Fractals	
Total	80

- The remaining time (20 hours) may be spent on additional practice, enrichment, or extension of the course.

Objectives

A. Probability

Foundational Objectives

- To demonstrate the ability to set up and calculate probabilities of related events (10 01 01). Supported by learning objectives 1 to 5.

- To apply the Binomial Theorem to expand binomials, and to real-world problems. (10 01 02). Supported by learning objectives 6, 7, and 8.

1. To define the principle of inclusion and exclusion when working with two or more sets and/or events.

2. To determine the probability of mutually exclusive events.

3. To determine the probability of two or more independent events.

4. To determine the probability of dependent events (conditional probabilities).

5. To set up, analyze, estimate, and solve word problems based on objectives 1 – 4.

6. To determine the coefficients of terms in a binomial expansion using the Binomial Theorem. (Pascal's Triangle r combinations could be used to introduce this topic).

7. To expand binomials of the form $(a+b)^n$ using the Binomial Theorem.

8. To solve word problems associated with Objectives 6 and 7.

B. Data Analysis

Foundational Objectives

- To determine the standard deviation of a set of data and to utilize the standard deviation in analyzing that set of data. (10 02 01). Supported by learning objectives 1 to 3.

- To develop skill in interpreting data through the use of z-scores (10 02 02). Supported by learning objectives 4 to 6.

1. To describe and illustrate normal and skewed distributions using real-world examples.

2. To calculate the standard deviation of a set of data.

3. To utilize the standard deviation to interpret data represented by a normal distribution.

4. To define and calculate z-scores.

5. To be able to utilize z-scores as an aid in interpreting data.

6. To solve related real-world problems using statistical inference.

C. Matrices

Foundational Objectives

- To illustrate appropriate real-world situations using matrices. (10 03 01). Supported by learning objective 2.

- To demonstrate knowledge in terms associated with matrices (10 03 02). Supported by learning objectives 1 and 6.

- To develop skills in matrix operations and in solving related real-world problems. (10 03 03). Supported by learning objectives 3, 4,5, 7, 8, 9, 10, 12.

1. To define basic terms associated with matrices.

2. To create a matrix to illustrate a given situation.

3. To add and subtract matrices.

4. To add and subtract matrices using scalar multiplication.

5. To multiply two matrices (not larger than 3 x 3).

6. To determine the properties of matrices with respect to addition, scalar multiplication, and multiplication.

7. To use row operations with matrices.

8. To determine the inverse of a "2x2" matrix.

9. To solve matrix equations using multiplication by an inverse. (Orders higher than 2x2 could be solved using technology).

10. To graph systems of inequalities.

11. To determine the points of intersection of lines drawn in objective 10.

12. To determine which vertices of the polygon formed by a system of inequalities maximizes or minimizes a given linear function.

D. Complex Numbers

Foundational Objective

- To demonstrate the skills developed in operations with complex numbers (10 04 01). Supported by the following learning objectives.

1. To define and illustrate complex numbers.

2. To express complex numbers in the form a + bi.

3. To add and subtract complex numbers.

4. To multiply and divide complex numbers.

5. To divide complex numbers using conjugates.

E. Quadratic Equations

Foundational Objectives

- To demonstrate skill in solving quadratic equations (10 05 01). Supported by learning objectives 1, 2, 3, 7, and 8.
- To write a quadratic equation through analysis of the given roots (10 05 02). Supported by learning objectives 4, 5, and 6.

1. To solve quadratic equations using the quadratic formula.

2. To solve quadratic equations having complex roots.

3. To solve word problems involving real-world applications of quadratic equations.

4. To determine the nature of the roots of a quadratic equation using the discriminant.

5. To determine that the sum of the roots of a quadratic equation $ax^2+bx+c= 0$ equals (-b/a) and the product of the roots equals (c/a).

6. To write a quadratic equation given the roots.

7. To solve equations of degree greater than two by expressing them in quadratic form. E.g.: $x^4-34x^2+225=0$.

8. To solve quadratic inequalities.

F. Polynomial and Rational Functions

Foundational Objectives

- To demonstrate the ability to graph and to analyze the graphs of polynomial and rational functions, (10 06 01). Supported by learning objectives 1 to 3.
- To demonstrate understanding of an inverse of a function (10 06 02). Supported by learning objectives 4 and 5.

1. To define and illustrate polynomial and rational functions.

2. To sketch the graphs of polynomial and rational functions with integral coefficients, using calculators or computers.

3. To analyze the characteristics of the graphs of polynomial and rational functions and to identify the 'zeros' of these graphs.

4. To define, determine, and sketch the inverse of a function, where it exists.

5. To define, determine, and sketch the reciprocal of a function.

G. Exponential and Logarithmic Functions

Foundational Objectives

- To develop skills and knowledge in working with a variety of exponential and logarithmic functions. (10 07 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to apply the knowledge of exponential and logarithmic functions to real-world situations. (10 07 02). Supported by learning objectives 6 to 17.

1. To define exponential functions and logarithmic functions.
2. To use correctly the laws of exponents for integral and rational exponents.
3. To work with logs of numbers with bases other than 10.
4. To construct graphs of exponential functions and logarithmic functions, to identify the properties of these graphs, and to recognize they are inverses of each other.
5. To sketch graphs o exponential and logarithmic functions by selecting an appropriate point for the new origin.
6. To solve exponential and logarithmic equations.
7. To solve word problems involving exponential and logarithmic functions.
8. To identify a geometric sequence.
9. To determine the nth term of a geometric sequence.
10. To calculate the required number of geometric means between given terms.
11. To calculate the sum of a geometric series.
12. To define and illustrate the following terms: geometric sequence, compound interest, present value, annuity, geometric means.
13. To determine the limit of a sequence.
14. To calculate the sum of an infinite series.
15. To solve word problems containing arithmetic or geometric series.
16. To solve word problems involving compound interest or present value.
17. To solve word problems involving annuities or mortgages.

Mathematics C 30

Major Concepts	**Number of Hours**
Mathematics C 30	
A. Mathematical Proof • Deductive proof • Indirect proof • Induction	23
B. Conic Sections • Circle • Parabola • Ellipse • Hyperbola • Systems of Equations	10 - 15
C. Circular Functions • Radian measure • Arc length • Graphs of trigonometric functions	15
D. Applications of Trigonometry • Laws of Sines and Cosines • Solving triangles, including the ambiguous case • Problems • Areas of triangles	12
E. Trigonometric Identities • Basic identities • Sum and difference identities • Double, half-angle, and n.θ identities	10
F. Trigonometric Equations • Particular and general solutions	5
* Optional Topics (objectives are not provided) • Fractals • Chaos Theory	
Total	75 - 80

- The remaining time (20 - 25 hours) may be spent on additional practice, enrichment, or extension of the course.

A. Mathematical Proof

Foundational Objective

- To appreciate the various types of mathematical thinking processes, and to demonstrate skill in applying these processes. (10 01 01). Supported by the following learning objectives.

  1. To define and illustrate by means of examples: deductive, inductive, and analogical statements or arguments.
  2. To complete deductive proofs from geometry, using a two-column format.
  3. To complete proofs using some of the methods of coordinate geometry.
  4. To use properties of numbers to justify solutions of alge-numeric exercises.
  5. To prove trigonometric identities using a two-column format.
  6. To introduce indirect proof and use it in several proofs.
  7. To introduce the principle of mathematical induction.
  8. To prove assertions by using mathematical induction.

B. Conic Sections

Foundational Objectives

- To become aware of the various conic sections and to demonstrate skill in graphing and writing equations of the conic sections (10 02 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to solve systems of linear-quadratic and quadratic-quadratic equations. (10 02 02). Supported by learning objectives 6 and 7.

  1a) To convert the equation of a circle from the general form to the standard form and vice versa.

  1b) To sketch the graph of a circle.

  2a) To convert the equation of a parabola from the general form to the standard form and vice versa.

  2b) To sketch the graph of a parabola.

  3a) To convert the equation of an ellipse from the general form to the standard form and vice versa.

  3b) To sketch the graph of an ellipse.

  4a) To convert the equation of a hyperbola from the general form to the standard form and vice versa.

  4b) To sketch the graph of a hyperbola.

  5. To examine the coefficients of the second-degree equations $Ax^2+By^2+Cx+Dy+E=0$ and identify the conic section it represents.
  6. To sketch diagrams to show possible relationships and intersections of the following systems: Linear-Quadratic and Quadratic-Quadratic.
  7. To solve the following systems of equations algebraically: Linear-Quadratic and Quadratic-Quadratic.

C. Circular Functions

Foundational Objectives

- To demonstrate an understanding of trigonometric functions as developed by circular functions (10 03 01) Supported by learning objectives 1 to 5.
- To be able to produce graphs of trigonometric functions. (10 03 02). Supported by learning objectives 5 and 7.

  1. To define the trigonometric functions and real numbers by wrapping a number line around a circle.
  2. To determine values of the primary and reciprocal trigonometric ratios.

3. To determine the radian measures of angles, to convert from radians to degrees and vice versa.

4. To determine angular velocity and to apply this concept to solving problems involving rotation.

5. To determine arc length and to apply this in associated problems.

6. To define and illustrate the following terms: periodic function, amplitude, domain, range, minimum value, maximum value, translation, wave motion, sinusoidal functions.

7. To state the range, period, amplitude, phase shift, minimum and maximum values and to sketch the graphs of:
   a) $y - k = a\sin(x - h)$
   b) $y - k = a\cos(x - h)$
   c) $y - k = a\tan(x - h)$

D. Applications of Trigonometry

Foundational Objectives

- To demonstrate the ability to apply trigonometry to real-world problem situations (10 04 01). Supported by learning objectives 1 to 5.

- To demonstrate the ability to calculate areas of given triangles using trigonometry (10 04 02). Supported by learning objectives 6 and 7.

1. To define and illustrate the following terms: angles of elevation and depression, heading, bearing, compass direction.

2. To solve right triangles and associated word problems.

3. To solve oblique triangles by the use of the Law of Sines/Cosines.

4. To solve triangles including all solutions given two sides and a non-included angle (the Ambiguous Case).

5. To solve word problems by means of the Law of Sines/Cosines.

6. To solve word problems by means of the law of Sines/Cosines.To determine the area of a triangle using $k = \frac{1}{2}ab\sin C$, $K = \frac{a^2 \sin B \sin C}{2\sin A}$, or Heron's formula $K=\sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of the triangle.

7. To solve word problems involving objective 6.

E. Trigonometric Identities

Foundational Objective

- To demonstrate the ability to work with trogonometric identities and to be able to apply them when necessary. (10 05 01). Supported by the following learning objectives.

1. To prove and apply the reciprocal identities.

2. To prove and apply quotient identities.

3. To prove and apply the Pythagorean identities.

4. To prove and apply the Addition/Subtraction identities.

5. To prove and apply the Double-Angle identities.

6. To determine $\sin n\theta$, where n is a natural number

7. To apply the Half-Angle identities.

F. Trigonometric Equations

Foundational Objective

- To demonstrate understanding and ability in solving trigonometric equations (10 06 01). Supported by the following learning objective.

1. To solve trigonometric equations by finding a particular equation and by finding the general solution.

Flowchart for Secondary Mathematics (Core): Strands and Concepts

Secondary Strands	Data/ Consumer	Number Operations	Equations Problems	Algebra	Functions	Geometry	Trigonometry
Math C 30		Mathematical Proof	Trigonometry Equations	Trigonometry Identities	Circular Functions	Conic Sections	Trigonometry Applications
Math B 30	Data Analysis	Complex Numbers Probability	Quadratic Equations	Matrices Application of Functions	Exponential & Logarithmic Functions Polynomial & Rational Functions		
Math A 30	Data Analysis	Exponents & Radicals Permutations & Combinations	Quadratic Equations Systems Linear Equations	Polynomials & Rational Expressions	Relations & Functions	Angles & Polygons	6 Trigonometry* Ratios (x,y,r)
Math 20	Consumer • Credit • Taxes	Irrational Numbers Probability	Quadratic Equations	Polynomials & Rational Expressions	Quadratic Functions	Angles Polygons Circles	Similarity* Scale Factor
Math 10	Consumer • Income • Budget		Equations Inequalities 1 Variable Linear	Algebra Numbers Polynomials	Relations Functions	Angles Polygons Lines Line Segments	Pythagorean Theorem* Primary Trigonometry Ratios
Middle Level Strands	Data Management	Numbers Operations	Problem Solving	Algebra	Ratio Proportion	Geometry/Measurement	

*Not a separate unit

Scope and Sequence for Secondary Mathematics (by strand)

The code in the column under the course of study refers to the Concept (capital letter) and the specific learning objective (number) under the Concept in that course of study.

For example, if the code C.6 appears in the 10 column under the course of study, this would indicate that it is in the Math 10 curriculum guide, the sixth learning objective in Concept C.

A ✓ in the Grade 9 column indicates this topic has been introduced in the middle level program.

Strand: Data Analysis/Consumer Mathematics

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
1. To calculate weekly gross wages involving regular pay, overtime pay, and piecework earnings.		C.1				
2. To calculate earnings for straight commission, or base wage plus commission.		C.2				
3. To determine the difference between gross pay and net pay.		C.3				
4. To calculate weekly, monthly, and yearly net pay.		C.4				
5. To define and explain the purpose of a budget.		C.5				
6. To determine and calculate monthly fixed expenditures.		C.6				
7. To investigate the guidelines in developing a budget.		C.7				
8. To plan a budget based on percentages allotted to various categories as suggested by financial institutions.		C.8				
9. To calculate the portion of total income spent on each category using percents.	✓	C.9				
10. To draw graphs (including circle graphs) representing budget figures, using appropriate software.	✓	C.10				
11. To calculate the actual amount of money to be spent on each category using the predetermined percentages.	✓	C.11				
12. To adjust a budget to changes in expenses.		C.12				
13. To define credit and determine its appropriate use.			B.1			

Strand: Data Analysis/Consumer Mathematics

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
14. To compare the advantages and disadvantages of various credit cards.			B.2			
15. To calculate the monthly interest charges and service charges on an unpaid credit card balance.			B.3			
16. To identify and compare an instalment charge account and a thirty-day account.			B.4			
17. To identify the characteristics of a personal loan.			B.5			
18. To compare the cost of a consumer loan from various institutions.	✓		B.6			
19. To calculate the monthly payments for a loan; using formulas, tables, calculators, and computers.	✓		B.7			
20. a) To determine the total cost of a personal loan. b) To determine the percent of the total amount repaid (or borrowed) which is devoted to interest.	✓		B.8			
21. To determine the importance of a credit rating.			B.9			
22. To describe how to establish a good credit rating.			B.10			
23. To calculate mill rates and property taxes.			B.11			
24. To calculate discounts or penalties on taxes due, depending on when they are paid.			B.12			
25. To determine and calculate permissible deductions from total income.			B.13			
26. To calculate the tax payable on taxable income.			B.14			

Strand: Data Analysis/Consumer Mathematics

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
27. To list, and describe, the methods used to collect data.	✓			B.1		
28. To obtain data for real-world situations by using simulations (such as Monte Carlo simulations).				B.2		
29. To review the methods for determining the measures of central tendency.	✓			B.3		
30. To construct box and whisker plots from simulated data.				B.4		
31. To define, and utilize, the concept of percentiles (including the first, second, and third quartiles).				B.5		
32. To solve related problems using statistical inference.				B.6		
33. To describe and illustrate normal and skewed distributions using real-world examples					B.1	
34. To calculate the standard deviation of a set of data.					B.2	
35. To utilize the standard deviation to interpret data represented by a normal distribution.					B.3	
36. To define and calculate z-scores.					B.4	
37. To be able to utilize z-scores as an aid in interpreting data.					B.5	
38. To solve related real-world problems using statistical inference.					B.6	

Strand: Numbers and Operations

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
1. To list the sample space and events for a random experiment.	✓		F.1			
2. To calculate the experimental probability of simple events by performing repeated experiments.	✓		F.2			
3. To calculate the theoretical probability of an event, and the probability of its complement.			F.3			
4. To define and illustrate, by means of examples, the term absolute value.			A.1			
5. To express square root radicals as mixed radicals in simplest form.			A.2			
6. To add, subtract, multiply, and divide square root radicals.			A.3			
7. To rationalize monomial denominators.			A.4			
8. To apply the fundamental counting principles to determine the number of possibilities that exist in a given situation.	✓			A.1		
9. To determine the number of permutations of n objects, (nPn=n!).				A.2		
10. To determine the number of permutations of n different objects, taken r at a time, (nPr).				A.3		
11. To determine the number of permutations of n objects, not all different.				A.4		
12. To determine the number of permutations of n objects arranged in a circle				A.5		
12. To determine the number of combinations of n objects, taken r at a time.				A.6		
13. To determine the number of combinations formed from more than one subset.				A. 7		
14. To evaluate powers with rational exponents.				D.1		

Strand: Numbers and Operations

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
15. To apply the laws of exponents to simplify expressions involving rational exponents.				D.2		
16. To write exponential expressions in radical form.				D.3		
17. To simplify square root and cube root expressions.				D.4		
18. To write radical expressions in exponential form.				D.5		
19. To add, subtract, multiply, and divide square root and cube root expressions.				D.6		
20. To rationalize monomial and binomial denominators in radical expressions.				D.7		
21. To define the principle of inclusion and exclusion when working with two or more sets and/or events.					A.1	
23. To determine the probability of mutually exclusive events.					A.2	
24. To determine the probability of two or more independent events.					A.3	
25. To determine the probability of dependent events (conditional probabilities).					A.4	
26. To set up, analyze, estimate, and solve word problems based on objectives 1-5.					A.5	
27. To determine the coefficients of terms in a binomial expansion using the Binomial Theorem. (Pascal's Triangle or combinations could be used to introduce this topic.)					A.6	
28. To expand binomials of the form $(a+b)^n$, using the Binomial Theorem.					A.7	

Strand: Numbers and Operations

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
29. To solve word problems associated with objectives 6 and 7.					A.8	
30. To define and illustrate complex numbers.					D.1	
31. To express complex numbers in the form a+bi.					D.2	
32. To add and subtract complex numbers.					D.3	
33. To multiply and divide complex numbers.					D.4	
34. To divide complex numbers using conjugates.					D.5	
35. To define and illustrate by means of examples: deductive, inductive, and analogical statements or arguments.						A.1
36. To complete deductive proofs from geometry, using a two-column format.						A.2
37. To complete proofs using some of the methods of co-ordinate geometry.						A.3
38. To use properties of numbers to justify solutions to alge-numeric exercises.						A.4
39. To prove trigonometric identities, using a two-column format.						A.5
41. To introduce indirect proof and use it in several proofs.						A.6
42. To introduce the principle of mathematical induction.						A.7
43. To prove assertions by using mathematical induction.						A.8

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
1. To solve linear equations in one variable containing: a) variables on both sides; b) parentheses; c) fraction or decimal coefficients.	✓	A.1				
2. To solve a formula for an indicated variable.	✓	A.2				
3. To solve, graph, and verify linear inequalities in one variable.	✓	A.3				
4. To translate English phrases into mathematical terms and vice versa.	✓	A.4				
5. To solve real-world problems using various problem solving strategies.		A.5				
6. To solve quadratic equations by: a) factoring, and b) by taking the square roots of both sides of an equation.	✓		E.1			
7. To calculate the exact value of the length of a side of a right triangle using the Pythagorean Theorem.			E.2			
8. To solve word problems involving quadratic equations.			E.3			
9. To solve and verify radical equations containing one radicand.			E.4			
10. To solve problems that involve equations which contain radicals.			E.5			
11. To solve and verify the solutions of quadratic equations by factoring, completing the trinomial square, and using the quadratic formula.				D.8		
12. To solve and verify equations involving absolute value.				D.9		
13. To solve radical equations with two unlike radicands.				D.10		
14. To solve word problems involving radical equations.				D.11		

Strand: Equations, Problems

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
15. To solve and verify systems of linear equations in two unknowns by the following methods: graphic, substitution, and elimination.				F.1		
16. To solve linear systems in two unknowns that have rational coefficients and verify the solutions.						
17. To recognize the characteristics of linear equations in two variables with graphs that are inconsistent, consistent-dependent, or consistent-independent.				F.2 F.3		
18. To solve word problems involving linear systems in two variables.						
19. To solve quadratic equations using the quadratic formula.				F.4		
20. To solve quadratic equations having complex roots.					E.1	
21. To solve word problems involving real- world applications of quadratic equations.					E.2	
22. To determine the nature of the roots of a quadratic equation using the discriminant.					E.3	
23. To determine that the sum of the roots of a quadratic equation $ax^2+bx+c=0$ equals (-b/a), and that the product of the roots equals (c/a).					E.4 E.5	
24. To write a quadratic equation, given the roots.					E.6	
25. To solve equations of degree greater than two by expressing them in quadratic form. eg. $x^4-34x^2+225=0$.					E.7	
26. To solve quadratic inequalities.					E.8	
27. To solve trigonometric equations by finding a particular solution and by finding the general solution.						F.1

Strand: Algebra

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
1. Numbers and Operations		F.1				
c) To add and subtract rational numbers.	✓					
d) To multiply and divide rational numbers.	✓					
e) To use the order of operations in evaluating rational arithmetic expressions.	✓					
2. Exponents		F.2				
a) To evaluate a positive power of a numerical base.	✓					
b) To evaluate multiplication and division of positive powers of the same numerical base.	✓					
c) To perform the product and quotient exponent properties with variable bases.	✓					
d) To write numbers in scientific notation and vice versa.	✓					
e) To perform multiplication and division of numbers expressed in scientific notation.	✓					
3. Polynomials		F.3				
a) To add and subtract polynomials.	✓					
b) To multiply a monomial by a monomial.	✓					
c) To multiply a polynomial by a monomial.	✓					
d) To multiply a binomial by a binomial.	✓					
e) To divide a monomial by a monomial.	✓					
f) To divide a polynomial by a monomial.						
4. To factor polynomials of the following types: common factor, grouping, difference of squares, trinomial squares, trinomials where a=1, trinomials where $a \neq 1$, and combinations of all preceding types.			C.1			

Strand: Algebra

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
5. To divide a polynomial by a binomial by factoring and by long division.			C.2			
6. To evaluate powers with positive and negative exponents.			C.3			
7. To simplify variable expressions with integral exponents using the following properties of exponents: product, quotient, power of a product, power of a quotient, negative exponent, and zero exponent.			C.4			
8. To determine the non-permissible values for the variable in rational expressions.			C.5			
9. To simplify rational expressions by factoring.			C.6			
10. To multiply and divide rational expressions.			C.7			
11. To add and subtract rational expressions involving like and unlike **monomial** denominators.			C.8			
12. To factor the difference of squares of special polynomials.				C.1		
13. To factor the sum and difference of cubes.				C.2		
14. To factor polynomials using the factor theorem.				C.3		
15. To use the remainder theorem to determine the remainder when a polynomial is divided by (x-r).				C.4		
16. To simplify rational expressions involving opposites.				C.5		
17. To add and subtract rational expressions with polynomial denominators.				C.6		
18. To multiply and divide rational expressions involving opposites.				C.7		

Strand: Algebra

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
19. To solve and verify linear equations in one variable involving rational algebraic expressions (including polynomial denominators).				C.8		
20. To solve and verify the solutions of quadratic equations involving rational algebraic expressions.				C.9		
21. To define basic terms associated with matrices.					C.1	
22. To create a matrix to illustrate a given situation.					C.2	
23. To add and subtract matrices.					C.3	
24. To add and subtract matrices using scalar multiplication.					C.4	
25. To multiply two matrices (not larger than 3x3).					C.5	
26. To determine the properties of matrices with respect to addition, scalar multiplication, and multiplication.					C.6	
27. To use row operations with matrices.					C.7	
28. To determine the inverse of a '2x2' matrix.					C.8	
29. To solve matrix equations using multiplication by an inverse. (Orders higher than 2x2 could be solved using technology.)					C.9	
30. To graph systems of inequalities.					C.10	
31. To determine the points of intersection of lines drawing in objective 10.					C.11	
32. To determine which vertices of the polygon formed by a system of inequalities maximizes or minimizes a given linear function.					C.12	

Strand: Algebra

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
33. To identify, describe, and interpret examples of direct variation in real-world situations.	✓	B.19				
34. To solve proportions involving direct variation.	✓	B.20				
35. To solve problems involving direct variation.	✓	B.21				
36. To identify partial variation.		B.22				
37. To solve problems involving partial variation.		B.23				
38. To define, illustrate, and identify an arithmetic sequence.		B.24				
39. To determine the n^{th} term of an arithmetic sequence.		B.25				
40. To define arithmetic means, and to determine the required arithmetic means between given terms.		B.26				
41. To calculate the sum of an arithmetic series.		B.27				
42. To determine the constant of proportionality of an inverse relation.				E.11		
43. To solve problems that involve inverse variation.				E.12		
44. To identify a geometric sequence.					G.8	
45. To determine the n^{th} term of a geometric sequence.					G.9	
46. To calculate the required number of geometric means between given terms.					G.10	
47. To calculate the sum of a geometric series.					G.11	

Strand: Algebra

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
48. To define and illustrate the following terms: geometric sequence, compound interest, present value, annuity, geometric means.					G.12	
49. To determine the limit of a sequence.					G.13	
50. To calculate the sum of an infinite series.					G.14	
51. To solve word problems containing arithmetic or geometric series.					G.15	
52. To solve word problems involving compound interest or present value.					G.16	
53. To solve word problems involving annuities or mortgages.					G.17	
54. To prove and apply the reciprocal identities.						E.1
55. To prove and apply the quotient identities.						E.2
56. To prove and apply the Pythagorean identities.						E.3
57. To prove and apply the Addition/Subtraction identities.						E.4
58. To prove and apply the Double-Angle identities.						E.5
59. To determine sin $n\theta$, where n is a natural number.						E.6
60. To apply the Half-Angle identities.						E.7

Strand: Functions

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
1. To define the following terms: relation, ordered pair, abscissa, ordinate.	✓	B.1				
2. To identify and express examples of relations in the real world.	✓	B.2				
3. To graph ordered pairs in the Cartesian coordinate plane and to graph real-world relations in the coordinate plane.	✓	B.3 B.4				
4. To read information from a graph.	✓	B.5				
5. To define the following terms: function, linear functions, slope, x-intercept, y-intercept, ratio, proportion, direct variation, partial variation.		B.1(b)				
6. To identify, graph and interpret examples of linear functions describing real-world situations.	✓	B.6				
7. To graph a linear function using a table of values.	✓	B.7				
8. To determine if a relation is a function by employing the vertical line test.		B.8				
9. To solve equations in two variables given the domain of one of the variables.		B.9				
10. To determine if an ordered pair is a solution to the linear equation.	✓	B.10				
11. To calculate the slope of a line: a) graphically (m = rise/run) b) algebraically $\left(m = \frac{y_2 - y_1}{x_2 - x_1}\right)$ c) from the equation ($y = mx + b$)		B.11				
12. To determine the slope of horizontal lines, vertical lines, parallel lines, and perpendicular lines.		B.12				
13. To write linear equations in a) slope-intercept form b) standard form		B.12(b)				

Strand: Functions

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
14. To graph a linear equation in two variables using b) x and y intercepts c) the slope and an ordered pair d) the slope and y-intercept (y=mx+b)		B.13,14				
15. To write the equation of a line when given a) slope and y-intercept b) slope and one point on the line c) the graph of the line d) two pints on the line		B.15,16				
16. To construct scatterplots from real-world data.		B.17				
17. To interpret and critically analyze these constructed scatterplots.		B.18				
18. To define a quadratic function.			D.1			
19. To identify, graph, and determine the properties of quadratic functions of the following forms: $f(x)=ax^2$, $f(x)=x^2+q$, $f(x)=(x-p)^2$, $f(x)=a(x-p)^2+q$.			D.2			
20. To determine the domain and range from the graph of a quadratic function.			D.3			
21. To analyze the graphs of quadratic functions that depict real-world situations.			D.4			
22. To solve problems involving the graphs of quadratic functions that depict real-world situations.			D.5			
23. To evaluate functions using functional notation.	✓			E.1		
24. To perform indicted operations on functions using function notation.	✓			E.2		
25. To form the composite of two or more given functions.				E.3		
26. To determine if a function is one-to-one or many-to-one				E.4		

Strand: Functions

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
27. To write the equation of a line in standard form using: two intercepts, slope and one intercept, one point ad the equation of a parallel line, and one point and the equation of a perpendicular line.				E.5		
30. To calculate the distance between two ordered pairs in the coordinate plane.				G.3		
31. To determine the coordinates of the midpoint of a segment.				G.4		
36. To solve real-world problems involving quadratic functions by analyzing their graphs.				E.6		
37. To graph quadratic functions of the standard form $f(x)=a(x-p)^2+q$, by determining the vertex, axis of symmetry, concavity, maximum or minimum values, domain, range, and zeroes.				E.7		
38. To graph quadratic functions of the general form $f(x)=ax^2+bx+c$, by completing the trinomial square and converting to one of the standard forms.				E.8		
39. To identify and graph examples of inverse variation taken from real-world situations.				E.9		
40. To state the domain and range, along with any restrictions, for the graphs of inverse variations				E.10		

Strand: Functions

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
41. To define and illustrate polynomial and rational functions.					F.1	
42. To sketch the graphs of polynomial and rational functions with integral coefficients, using calculators or computers.					F.2	
43. To analyze the characteristics of the graphs of polynomial functions, and to identify the 'zeros' of these graphs.					F.3	
44. To define, determine, and sketch the inverse of a function, where it exists.					F.4	
45. To define, determine, and sketch the reciprocal of a function.					F.5	
46. To define exponential functions and logarithmic functions.					G.1	
47. To use correctly the laws of exponents for integral and rational exponents.					G.2	
48. To work with logs of numbers with bases other than 10.					G.3	
49. To construct graphs of exponential functions and logarithmic functions, to identify the properties of these graphs, and to recognize they are inverses of each other.					G.4	
50. To sketch graphs of exponential and logarithmic functions by selecting an appropriate point for the new origin.					G.5	
51. To solve exponential and logarithmic equations.					G.6	
52. To solve word problems involving exponential and logarithmic functions.					G.7	

Strand: Functions

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
53. To define the trig functions of real numbers by wrapping a number line around a circle.						C.1
54. To determine values of the primary and reciprocal trigonometric ratios.						C.2
55. To determine the radian measures of angles, to convert from radians to degrees and vice versa.						C.3
56. To determine angular velocity and apply this concept in solving problems involving rotation.						C.4
57. To determine arc length and apply this in associated word problems.						C.5
58. To define and illustrate the following terms: period function, amplitude, domain, range, minimum value maximum value, translation, wave motion, sinusoidal functions.						C.6
59. To state the range, period, amplitude, phase shift, minimum and maximum values, and sketch the graphs of: a) $y - k = a \sin b(x-h)$ b) $y - k = a \cos b(x-h)$ c) $y - k = a \tan b(x-h)$						C.7

Strand: Geometry

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
1. To define: line segment, ray, line, bisector, perpendicular line, perpendicular bisector, transversal, alternate interior angles, corresponding angles, same-side interior angles.	✓	D.1				
2. To identify and calculate the measures of the following angles formed by parallel lines, corresponding angles, alternate interior angles, and same-side interior angles.	✓	D.2				
3. To solve word problems involving angles formed by parallel lines.	✓	D.3				
4. To informally and formally construct: a) congruent segments; b) the perpendicular bisector of a line segment; c) a line perpendicular to a given line from a point not on the line; d) a line perpendicular to a given line from a point on the line; and e) a line parallel to a given line through a point not on the line.	✓	D.4,5, 6,7				
5. To define and illustrate by drawing the following: acute angle, right angle, obtuse angle, straight angle, reflex angle, complementary angles, supplementary angles, adjacent angles, vertically opposite angles, congruent angles, central angles of a regular polygon.	✓	E.1				
6. To solve word problems involving the angles stated in E.1.	✓	E.2				
7. To define and illustrate the following polygons: convex, non-convex, regular, quadrilateral parallelogram, rectangle, rhombus, square, trapezoid, isosceles trapezoid.		E.3(a)				
8. To define and illustrate the following triangles: scalene, isosceles, equilateral, acute, right, and obtuse.		E.3(b)				

Strand: Geometry

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
11. To classify quadrilaterals as trapezoids, isosceles trapezoids, parallelograms, rectangles, rhombuses, and squares.	✓	E.4				
12. To informally construct parallelograms, rectangles, rhombuses, and squares.	✓	E.5				
13. To state and apply the properties of parallelograms a) opposite sides are parallel b) opposite sides are congruent c) opposite angles are congruent d) the diagonals bisect each other	✓	E.6				
14. To determine the sum of the measures of the interior and exterior angles of a convex polygon of n sides.		E.7				
15. To determine the measure of a central angle in a regular n-gon.		E.8				
16. To determine the measures of the interior and exterior angles of regular n-gons.		E.9				
17. To determine the number of diagonals in a polygon of n sides.		E.10				
18. To define the measure of a minor arc, and to calculate the measure of a central angle.			H.1			
19. To determine the relationship that exists between the following: - the radius of a circle and a tangent line drawn to it at the point of tangency; - two tangents drawn to a circle from the same point; - chords and arcs in the same circle or in congruent circles; - a diameter and a chord bisected by the diameter; and, - two chords that intersect inside a circle.			H.2			
20. To solve problems based on the relationships state in H.2.	✓		H.3			
21. To informally and formally construct congruent angles, and congruent triangles.			G.1			

Strand: Geometry

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
22. To determine the properties of congruent triangles.	✓		G.2			
23. To identify and state corresponding parts of congruent triangles.	✓		G.3			
24. To determine whether triangles are congruent by SSS, SAS, ASA, AAS, or HL.	✓		G.4			
25. To prove that two triangles are congruent by supplying the statements and reasons in a guided deductive proof.	✓		G.5			
26. To prove triangles congruent by SSS, SAS, AAS, ASA, or HL in a two-column deductive proof or paragraph form.			G.6			
27. To prove corresponding parts of congruent triangles are congruent.			G.7			
28. To convert the equation of a circle from the general form to the standard form and vice versa.						B.1(a)
29. To sketch the graph of a circle.						B.1(b)
30. To convert the equation of a parabola from the general form to the standard form and vice versa.						B.2(a)
31. To sketch the graph of a parabola.						B.2(b)
32. To convert the equation of an ellipse from the general form to the standard form and vice versa.						B.3(a)
33. To sketch the graph of an ellipse.						B.3(b)
34. To convert the equation of a hyperbola from the general form to the standard form and vice versa.						B.4(a)
35. To sketch the graph of a hyperbola.						B. 4(b)

Strand: Geometry

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
36. To examine the coefficients of the second degree equation $Ax^2+By^2+Cx+Dy+E=0$, and identify the conic section it represents.						B.5
35. To sketch diagrams to show possible relationships and intersections of the following systems: Linear-Quadratic; and Quadratic-Quadratic.						B.6
36. To solve the following systems of equations algebraically: Linear-Quadratic and Quadrati-Quadratic.						B.7

Strand: Trigonometry

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
1. To calculate to two decimal places the length of a missing side of a right triangle using the Pythagorean Theorem.	✓	E.11				
2. To solve word problems using the Pythagorean Theorem.	✓	E.12				
3. To determine if a triangle is a right triangle by using the converse of the Pythagorean Theorem.		E.13				
4. To determine the value of the three primary trigonometric ratios using a calculator.		E.14				
5. To determine the measure of an angle given the value of one trigonometric ratio of the angle by using a calculator.		E.15				
6. To calculate the measure of an angle or the length of a side of a right triangle using the tangent, sine and cosine ratios.		E.16				
7. To solve word problems that involve trigonometric ratios using a calculator.		E.17				
8. To identify similar polygons.	✓		G.8			
9. To determine the measure of corresponding angles in two similar polygons.	✓		G.9			
10. To calculate the scale factor of two similar polygons.	✓		G.10			
11. To calculate the length of a missing side of two similar polygons.			G.11			
12. To show that two triangles are similar by the Angle Angle Similarity Theorem (Postulate in some resource texts).			G.12			
13. To calculate the length of a missing side in two similar right triangles.			G.13			
14. To solve problems involving similar triangles and other polygons.			G.14			

Strand: Trigonometry

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
15. To determine surface area and volumes of similar polygons or solids.	✓		G.15			
16. To sketch an angle in standard position.				G.1		
17. To determine the distance from the origin to a point on the terminal arm of an angle in standard position.				G.2		
18. To determine the value of the six trigonometric ratios when given a point on the terminal arm of an angle in standard position (x,y,r).				G.5		
19. To determine coterminal angles for a given angle.				G.6		
20. To determine the reference angle for positive or negative angles.				G.7		
21. To determine the values of the six trigonometric ratios, when given one trigonometric ratio and the quadrant in which the angle terminates.				G.8		
22. To determine the values for the trigonometric ratios by using a calculator.				G.9		
23. To apply the trigonometric ratios to problems involving right triangles.				G.10		
24. To determine the relationships among the sides of each special right triangle (45-45-90 and 30-60-90).				G.11		
25. To calculate the length of the missing sides of the special right triangles when given the exact value of one side.				G.12		
26. To define and illustrate the following terms: angles of elevation and depression, heading, bearing, compass direction.						D.1
27. To solve right triangles and associated word problems.						D.2

Strand: Trigonometry

Learning Objectives	Course of Study					
	9	10	20	A 30	B 30	C 30
28. To solve oblique triangles by the use of the Law of Sines/Cosines.						D.3
29. To solve triangles including all solutions given two sides and a non-included angle (the Ambiguous Case).						D.4
30. To solve word problems by means of the Law of Sines/Cosines.						D.5
31. To determine the area of a triangle using $K = ½ ab \sin C$, $K = \frac{a^2 \sin B \sin C}{2 \sin A}$, or Heron's Formula $K = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter of the triangle.						D.6
32. To solve word problems involving Objective D6.						D.7

Templates for Assessment and Evaluation

The templates included represent some examples of the various types of evaluation and assessment the teacher might employ. The teacher is encouraged to peruse *Student Evaluation: A Teacher Handbook* (1991) and curriculum guides from other subject areas in order to obtain a more comprehensive set of evaluation templates. In addition, the teacher may adapt any of the templates to accommodate students in the classroom.

Mathematics 30

Algebraic Skills

Student	Numbers and Operation				Exponents				Polynomials														

Rating Scale

Activity: Problem Solving
Course: Mathematics 30

Date:

Student	Description of Activity Component	Scale Points				
	1. Understands the Problem					
	2. Devises a Plan					
	3. Executes the Plan					
	4. Reflects					
	5.					
	1.					
	2.					
	3.					
	4.					
	5.					
	1.					
	2.					
	3.					
	4.					
	5.					
	1.					
	2.					
	3.					
	4.					
	5.					

Rating Scale

Activity:
Course:

Date:

Student	Description of Activity Component	Scale Points				
	1.					
	2.					
	3.					
	4.					
	5.					
	1.					
	2.					
	3.					
	4.					
	5.					
	1.					
	2.					
	3.					
	4.					
	5.					
	1.					
	2.					
	3.					
	4.					
	5.					

GRADE: Mathematics 30 **TOPIC:** Polynomial Functions **DATE:**

A - able

U - unable

	Define	Identify	Generate Examples	Interpret	Graph			COMMENTS

GRADE: TOPIC: DATE:

GRADE: **DATE:**

TOPIC: Communicating Mathematically (oral and written)

	Clarify	Accuracy	Articulation	Proper Symbols	Proper Terms					

GRADE: TOPIC: DATE:

Observational Rating Scale

Group Work

Topic:

The student:	Seldom - 1 Always - 4	Comments
1. Volunteers information or ideas		
2. Shows willingness to listen		
3. Asks good questions		
4. Considers facts		
5. Shows respect for others		
6. Supports ideas with facts		

Teacher Notes:

	Seldom - 1 Always - 4	Comments

Teacher Notes:

Grade: Mathematics 10
Topic: Budget
Date: June 3

Student	Rating 1 - Fair 2 - Good 3 - Very Good 4 - Excellent	Comments
Corry	2	Understands the concept of budgeting but has difficulty working with percent when calculating portion of total income spent on categories.
Melissa	4	Fully understands all aspects of budgeting. Is able to plan, display, adjust and interpret a budget.

Grade:
Topic:
Date:

Student	**Rating** 1 - Fair 2 - Good 3 - Very Good 4 - Excellent	**Comments**

Anecdotal Records

Grade: Mathematics 10
Activity: Angles and Polygons

Date: May 20-24

Tom Is unable to identify vertically opposite angles in a diagram.				Jim Uses a variety of methods when calculating the measure of the angles and sides of a right triangle.
		Lori Works quickly and accurately when calculating the measure of complementary and supplementary angles.		
				Donna Confuses a central angle with an interior angle of a polygon.
Patti Can not remember and identify the various kinds of angles.		Brent Uses incorrect Trig. ratio when solving word problems.		
			Jason Has difficulty identifying the hypotenuse in a right triangle when the triangle is rotated.	
	Shannon Recognizes parallelograms but is unable to determine if one parallelogram has the properties of another.			

Anecdotal Records

Grade: **Date:**
Activity:

Individual Group Evaluation Form

We will stop once weekly to hold discussion in your groups about the process of working together. This is a time to consider how you feel and what you think about working in your group. Thinking about the process of working together helps people to recognize strengths and to become aware of ways in which we can improve our working relationships.

Please answer the following three questions on your own. Then use the three questions and your response as the basis for discussion within your group.

1. How do you feel about your participation as a member of your group at this time?

   Please circle:

   Very Satisfied   Quite Satisfied   Somewhat Dissatisfied   Quite Dissatisfied

   Please comment on why you checked where you did:

   ____

   ____

   ____

2. How do you feel about the productivity of your group at this time?

   Please circle:

   Very Satisfied   Quite Satisfied   Somewhat Dissatisfied   Quite Dissatisfied

   Please comment on why you checked where you did:

   ____

   ____

   ____

3. What things might I, or we, do to improve our group functioning as we continue to work?

   ____

   ____

   ____

Clarke, J., Wideman, R., Eadie, S. (1990). *Together we learn: Cooperative small group learning*. Scarborough: Prentice-Hall Canada Inc. p. 106. Reproduced with permission.

Observation Form

Place check marks in the appropriate boxes as you watch and listen. You might also want to record a few examples of what group members do or say.

	Asking Questions	Seeking Information and Opinions	Responding to Ideas	Acknowledging Contributions
Group 1				
Group 2				
Group 3				
Group 4				
Group 5				

Date ____________________

Clarke, J., Wideman, R., Eadie, S. (1990). *Together we learn: Cooperative small group learning.* Scarborough: Prentice-Hall Canada Inc. p. 131. Reproduced with permission.

Sample Criteria Checklist Developed by the Teacher and Students for Summative Peer and Teacher Evaluation

Criteria for Effective Group Presentation to Review Concepts in Trigonometry

The group:	Not at all				Thoroughly
• appeared prepared and organized.	___	___	___	___	___
• was knowledgeable about its section.	___	___	___	___	___
• worked together as a group.	___	___	___	___	___
• encouraged active participation from the class.	___	___	___	___	___
• demonstrated patience and helpfulness.	___	___	___	___	___
• used a variety of teaching techniques.	___	___	___	___	___

One part of the presentation which was particularly helpful (and why):

One suggestion for improvement:

Clarke, J., Wideman, R., Eadie, S. (1990). *Together we learn: Cooperative small group learning*. Scarborough: Prentice-Hall Canada Inc. p. 138. Adapted.

Sample Evaluation of Another Group's Presentation

- How did the group capture your interest in the topic?

- Describe three things you learned about the topic from the presentation.

- Describe one thing about the presentation that you thought was creative or imaginative.

- Make one suggestion which you think would strengthen the effectiveness of the presentation.

- Was there anything about the presentation that makes you interested in learning more about the topic? If so, please indicate what it was.

Names ____

Clarke, J., Wideman, R., Eadie, S. (1990). *Together we learn: Cooperative small group learning*. Scarborough: Prentice-Hall Canada Inc. p. 160. Reproduced with permission.

Unit Planning

There are many ways to plan a unit. The following unit planning guide and planning sheet are not prescriptive. They offer some suggestions to teachers so that they may plan for the maximum benefits for their students. In any case, all planning should include expected learning objectives outcomes, instructional strategies, evaluation strategies, and the Adaptive Dimension.

Unit Planning Guide

1. Determine the sequence of the concepts that are to be developed.

2. Examine the Foundational Objectives with the subsequent learning objectives. Include the development of the Common Essential Learnings.

3. Determine the prerequisite skills required.

4. Identify the print and non-print resources to meet the needs of the students. Consider community resources. Access available electronic resources.

5. Develop activities that are appropriate for the objectives.

6. Consider a variety of instructional strategies and methods for the activities. Select those that are most appropriate in meeting the objectives and in meeting the learning styles and needs of the students.

7. Determine the organizational structures that support the instructional methods and activities to be used.

8. Consider how the activity might be linked to other areas of study. Modify the activity to strengthen the connections.

9. Analyze how the Common Essential Learnings can be developed within the activities of each lesson.

10. Consider the initiatives of Gender Equity, Indian and Métis Perspectives, Inclusionary Education, and Agriculture in the Classroom. How can they be incorporated into the unit?

11. Adapt for individual differences whether it be curriculum topics and materials, instruction, or environment (Adaptive Dimension).

12. Select student evaluation strategies. See the section in this guide and consult *Student Evaluation: A Teacher Handbook* (1991). Just as a variety of activities should be chosen to accomplish the objectives, a variety of evaluation strategies should be employed so that all aspects of learning can be assessed.

Note: Teachers are challenged to create a formal unit by integrating the Mathematics C 30 section on Concept A: Mathematics Proof with Appendices G, H, and I -- student and teacher resources.

Planning Sheet

Unit Title:

Foundational Objectives	Specific Learning Objective Verbs	CELs	Instructional Strategy and/or Method	Resources	Assessment Technique	Adaptations
				Human Manipulatives Print Non-Print (e.g., software, video) Technology Other Subject Areas		

Direct Instruction
Structured Overview
Explicit Teaching
Mastery Lecture
Drill and Practice
Compare and Contrast
Didactic Questions
Demonstrations
Guides for Reading, Listening, Viewing

Experiential Learning
Field Trips
Conducting Experiments
Simulations
Games
Focused Imaging
Field Observations
Role Playing
Synectics
Model Building Surveys

Independent Learning
Essays
Computer Assisted Instruction
Reports
Learning Activity Package
Correspondence Lessons
Learning Contracts
Homework
Research Projects
Assigned Questions
Learning Centres

Indirect Instruction
Problem Solving
Case Studies
Inquiry
Reading for Meaning
Reflective Discussion
Concept Formation
Concept Mapping
Concept Attainment
Cloze Procedure

Interactive Instruction
Debates
Role Playing
Panels
Brainstorming
Peer Practice
Discussion
Laboratory Groups
Cooperative Learning Groups
Problem Solving
Circle of Knowledge
Tutorial Groups
Interviewing

Common Essential Learnings
Communication
Critical and Creative Thinking
Independent Learning
Numeracy
Technological Literacy
Personal and [illegible] Values and Skills

Methods of Organization
Assessment Stations
Individual Assessments
Group Assessments
Contracts
Peer and Self-Assessments
Portfolios

Methods of Data Recording
Anecdotal Records
Observation Checklists
Rating Scales

Ongoing Student Activities
Written Assignments
Presentations
Performance Assessments
Homework

Quizzes and Tests
Oral Assessment Items
Performance Assessment Items
Extended Open Response Items
Short Answer Items
Matching Items
Multiple Choice Items
True/False Items

Other Initiatives
Resource-based Learning
Indian & Métis Perspectives
Gender Equity
Practical and Applied Emphasis

References

The following materials were used as background during the preparation of this document:

Hembree, Ray & Dessart Donald J. (1992). Research on calculators in mathematics education. In James T. Fey (Ed.), *Calculators in mathematics education, 1992 yearbook* (pp. 23-32). Reston, VA: National Council of Teachers of Mathematics.

Hope, J. (1990). *Charting the course: A guide for revising the mathematics program in the province of Saskatchewan.* Saskatchewan Instructional Development and Research Unit (SIDRU) Report No. 16. Regina, SK: University of Regina.

Jensen, Robert J. & Williams, Brevard S. (1993). Technology: Implications for middle grades mathematics. In Douglas T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 225-243). Don Mills, ON: Maxwell Macmillan Canada, Inc.

Miles, D. D. and Forcht, J. P. (1995). Mathematics strategies for secondary students with learning disabilities or mathematics deficiencies: A cognitive approach. *Intervention in School and Clinic*, 31(2), 91-96.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Polya, G. (1971). *How to solve it*. Princeton, NJ: Princeton University Press.

Saskatchewan Education. (1992). *The adaptive dimension in core curriculum*. Regina, SK: Author.

Saskatchewan Education. (1991). *Instructional Approaches: A framework for professional practice*. Regina, SK: Author.

Saskatchewan Education. (1991). *Student evaluation: A teacher handbook*. Regina, SK: Author.

Saskatchewan Education. (1989). *Charting the course: Summary and recommendations*. Regina, SK: Author.

Saskatchewan Education. (1988). *Understanding the common essential learnings: A handbook for teachers*. Regina, SK: Author.

Saskatchewan Education. (1984). *A curriculum guide for division IV algebra 10, 20, 30 and geometry-trigonometry 10, 20, 30 (Interim Edition)*. Regina, SK: Author.

Saskatchewan Instructional Development and Research Unit/Saskatchewan Professional Development Unit (SIDRU/SPDU). (1990-1994). *Instructional strategies series*. Regina and Saskatoon, SK: University of Regina/Saskatchewan Teachers' Federation.

Western Canadian Protocol for Collaboration in Basic Education (Crown rights). (1995). *The common curriculum framework for K-12 mathematics* (10-12 under development). Alberta Education: Western governments. ISBN 0-7732-1780-0 (Note: the 10-12 document will be released in 1996.)

Mathematics A 30

Western Protocol - Common Curriculum Framework (1996)

10-12 Mathematics - General Outcomes

Number (Number Concepts)

- Analyze graphs or charts of given situations to derive specific information.
- Analyze the data in a table for trends, patterns and interrelationships.
- Explain and illustrate the structure and the interrelationship of the sets of numbers within the real number system.
- Explain and illustrate the structure of the complex number system and its subsets.

Number (Number Operations)

- Use basic arithmetic operations on real numbers to solve problems.
- Describe and apply arithmetic operations on tables to solve problems, using technology as required.
- Describe and apply arithmetic operations on matrices to solve problems, using technology as required.
- Make and justify financial decisions.

Patterns and Relations (Patterns)

- Represent naturally occurring discrete data, using linear or nonlinear functions.
- Generate and analyze number patterns.
- Investigate the nature of mathematical reasoning.
- Generate and analyze recursive and fractal patterns.

Patterns and Relations (Variables and Equations)

- Generalize operations on polynomials to include rational expressions.
- Represent and analyze situations that involve variables, expressions, equations and inequalities.
- Use linear programming to solve optimization models.
- Solve exponential, logarithmic and trigonometric equations.

Patterns and Relations (Relations and Functions)

- Examine the nature of relations with an emphasis on functions.
- Represent by models naturally-occurring data using linear functions.
- Represent and analyze functions using technology, as appropriate.
- Use the concept of function to solve problems.

Shape and Space (Measurement)

- Use measuring devices to make estimates and to perform calculations in solving problems.
- Demonstrate an understanding of scale factors, and their interrelationship with the dimensions of similar shapes and objects.
- Solve problems involving triangles, including those found in 3-D applications.
- Analyze objects, shapes, and processes to solve cost and design problems.

Shape and Space (3-D Objects and 2-D Shapes)

- Solve co-ordinate geometry problems involving lines and line segments.
- Develop and apply the geometric properties of circles and polygons to solve problems.
- Classify conic sections, using their shapes and equations.
- Solve problems involving triangles and vectors, including 3-D applications.

Shape and Space (Transformations)

- Perform, analyze and create transformations of functions and relations.

Statistics and Probability (Data Analysis)

- Describe, implement and analyze sampling procedures and draw appropriate inferences from the data collected, using mathematical and technical language.
- Apply line-fitting techniques to analyze experimental results.
- Analyze bivariate data.

Statistics and Probability (Chance and Uncertainty)

- Make and analyze decisions using expected gains and losses based on single events.
- Model the probability of a compound event in order to solve problems based on the combining of simpler probabilities.
- Solve problems based on the counting of sets, using techniques such as the fundamental counting principle, permutations and combinations.
- Use normal probability distribution to solve problems involving uncertainty.

Mathematics A 30

A. Permutations and Combinations

Foundational Objectives

- To demonstrate the ability to determine the number of permutations in a given situation (10 01 01)*. Supported by learning objectives 1 to 5.
- To demonstrate the ability to determine the number of combinations in a given situation (10 01 02). Supported by learning objectives 6 and 7.

B. Data Analysis

Foundational Objectives

- To demonstrate developed skills and understanding in collecting and displaying a set of data for a given situation (10 02 01). Supported by learning objectives 1 to 4.
- To provide reasonable explanations of the interpretation of a set of data (10 02 02). Supported by learning objectives 5 and 6.

C. Polynomials and Rational Expressions

Foundational Objectives

- To demonstrate ability in the addition, subtraction, multiplication, and division of rational expressions (10 03 01). Supported by learning objectives 1 to 7.
- To demonstrate ability in solving equations involving rational expressions (10 03 02). Supported by learning objectives 8 and 9.

D. Exponents and Radicals

Foundational Objectives

- To be able to illustrate the relationship between the radical and exponential forms of an equation (10 04 01). Supported by learning objectives 1, 3, and 5.
- To demonstrate the ability to work with operations involving radical numbers (10 04 02). Supported by learning objectives 2, 4, 6, and 7.
- To be able to solve equations involving radicals, and to be able to justify the solutions (10 04 03). Supported by learning objectives 8 to 11.

E. Relations and Functions

Foundational Objectives

- To demonstrate the ability to work with functional notation and related operations (10 05 01). Supported by learning objectives 1, 2, and 3.
- To be able to produce graphs of relations and functions, and to be able to denote which graphs represent functions (10 05 02). Supported by learning objectives 4 and 8.
- To demonstrate the ability to interpret graphs representing functions, and to identify key points of these graphs (10 05 03). Supported by learning objectives 5, 6, and 7.
- To be able to identify inverse variations, and to demonstrate solutions to problems involving inverse variations (10 05 04). Supported by learning objectives 9 to 12.

F. Systems of Linear Equations

Foundational Objectives

- To be able to identify the number of possible solutions of a system of linear equations (10 06 01). Supported by learning objective 3.
- To demonstrate the ability to solve a system of linear equations (10 06 02). Supported by learning objectives 1, 2, and 4.

G. Angles and Polygons

Foundational Objectives

- To demonstrate the ability to determine trigonometric ratios in a given situation, and apply these ratios to solving real-world problems (10 07 01). Supported by the following learning objectives.

* Saskatchewan Education uses codes (10 01 01) for assessment and evaluation purposes.

Concept A: Permutations and Combinations

Foundational Objectives

- To demonstrate the ability to determine the number of permutations in a given situation (10 01 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to determine the number of combinations in a given situation (10 01 02). Supported by learning objectives 6 and 7.

Objectives

A.1

To apply the fundamental counting principles to determine the number of possibilities that exist in a given situation.

Instructional Notes

It may be necessary to review some of the underlying principles of set theory before beginning the study of this section. The concepts of set, subset, and mutually exclusive should be understood, at least intuitively.

The fundamental counting principles are those which deal with the combining of the numbers of outcomes in given situations.
FCP 1 (The Multiplication Principle)

If the first part of a procedure can be performed in m ways, and the second part of the procedure can be performed in n ways, then the procedure can be carried out in mn ways.

FCP 2 (The Addition Principle)

If one task can be performed in m ways, and a second, mutually exclusive task can be performed in n ways, then the number of ways of performing either task is given by m+n ways.

Students should work in groups to determine the answers to several questions involving these counting principles. Discussion should be encouraged within the groups, so that group members are satisfied that their group answers are suitable. Group answers should be summarized. (PSVS)

Examples/Activities

Student groups could work on a series of questions similar to the following:

1. The Ajax Phone Company offers three basic models of telephone (princess, desk, and wall). Each model is available in green, beige, red, black, or white; and each can be purchased with either touch-tone, or automatic dialling. How many variations of phone does the Ajax company have ? What is the probability of randomly getting a green touch-tone phone?

2. A new radio station is to choose its call letters. There are to be four letters in all. The first letter must be a C, the second letter must be chosen from B, F, J, K, or S. As well, the radio station wishes to have a vowel as the third letter in its 'name'. How many choices does this station have? What if the station wishes to have exactly one vowel in its name? What is the probability it is randomly assigned two vowels?

3. How many numbers that are less than 10 000 can be formed if: a) the first digit cannot be zero, and b) no repetition of digits is allowed.

(Hint: You can use one, two, three, or four digit numbers.)

Adaptations

Student groups could be instructed to share some real-life instances where fundamental counting principles are used (e.g.; generation of postal codes, telephone numbers, types of hamburgers available at local booths or concessions, licence plates, etc.).

Some calculations might be done with these examples to determine if the needs of the community are being served by these systems. (E.g.: Will the present system of phone numbers be suitable if the community expands in the future? How could this be determined?)

The teacher may wish to introduce questions dealing with the intersection and union of sets at this point. These may be introduced using counting arguments or overlapping sets. A number of examples should be done to illustrate concepts such as the principles of inclusion and exclusion.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example: 93 students attended a school dance. 62 students purchased hotdogs while 45 purchased a soft drink, and 8 had neither. How many students had both a hotdog and a soft drink?

Concept A: Permutations and Combinations

Objectives	Instructional Notes
A.2 To determine the number of permutations of n objects. (nPn = n!)	Students should be given the definition of a permutation, and shown some examples of permutations (e.g; 3-letter nonsense syllables composed of the letters a, b, and c). The idea of a permutation of a complete set of elements and its relation to the multiplication principle should be explored by having groups work on selected examples or exercises. When students are comfortable with their understanding, the factorial notation could be introduced, and formalized. Calculators, or computers should be employed to do calculations of larger permutations, and perhaps to run simulations. Note the special case of 0! = 1. Have students try this on their calculators as well.
A.3 To determine the number of permutations of n different objects, taken r at a time. (nPr)	Students can be given some relatively simple examples to work on in their groups. The group answers should be shared, with ensuing discussion leading to a more uniform method of approaching these situations. (An example that they might work on is: list all the different two-letter syllables that could be formed from the word 'star'.) The formula should be developed, and further examples done to illustrate the use of the formula. Also, utilize calculators and computers.

Examples/Activities

Exercises similar to the following can be assigned to the groups. Solutions can be shared, and formalized for the entire class.

1. In how many different ways can we arrange the letters of the word 'draw'? What is the probability the arrangement begins with a vowel?

2. How many different 5-digit numbers can be formed from the digits 2, 3, 5, 7, 8, if each digit is used only once in each number? What is the probability the number ends in a 5?

3. In how many ways can we arrange eight different books on a shelf?

Students could be assigned a series of exercises to be done in groups. These might be similar to the following:

1. How many different nonsense syllables of three letters can be formed from the letters of the word 'groups', if no letter can be used more than once? What is the probability that both vowels appear in the nonsense syllable?

2. In how many ways can we arrange a group of four students chosen from a group of 10 students?

Adaptations

Students could be asked to generate permutations for their own groups to solve. Each example would have to be reflected upon to see if it fits the criteria of a permutation of a set of objects, before the calculations are carried out.

Students should use the multiplication principle to find the number of ways to form a queue of length n from n people. They would fill in the first position, then the second, and so on.

Students could be asked to determine real-world situations in which permutations of this type are encountered. Then the number of permutations of each example can be determined by using formula, calculator, or computer. The teacher should have some examples ready in case students are not able to generate their own examples. (CCT)

Students could be asked to find the number of ways to form a queue of length r from n people using the multiplication principle.

Students could be instructed to answer the following: If you could do one operation from a list of 20! every second, how many years would it take to perform all the operations ?

Concept A: Permutations and Combinations

Objectives	Instructional Notes
A.4 To determine the number of permutations of n objects, not all different.	Students could work in groups to determine how many different ways they could arrange three objects in a line, if all were different. They could then progress to the situation where two of the objects were identical. Extend this to have them determine the number of ways to arrange four objects in a line, if two were identical (then three identical). At this point, they could be asked to conjecture a result for the general case. Have them predict the result with five objects, two of which are identical. Extend this to more difficult situations, such as five objects, of which three are identical to each other, and the remaining two are identical to each other. Manipulatives, such as bingo chips, algebra tiles, dice, could be useful for this activity.
A.5 To determine the number of permutations of n objects arranged in a circle.	For circular permutations, students might be given a situation to explore individually, in pairs, or small groups. Use of manipulatives such as small figurines, coins, stamps, and the like, may be useful for some, but these could also be done by assigning alpha-numeric symbols to positions on a circle. The problem assigned might be to determine in how many ways three (then four, five, and n) objects can be arranged in a circle. Students should be instructed that there is no 'fixed' starting position, and to note each permutation. After determining the results for a few of these, students should be expected to determine the formula for this type of permutation. { (n-1)! } Some exercises that utilize this formula can be assigned for practice. The special case of the 'key-ring' type of permutation can also be dealt with in this section. Because some objects, such as coloured beads on a necklace, or keys, can be arranged in a circle (key-ring) without regard to whether they are right side up, the number of permutations is given by the formula $\frac{(n-1)!}{2}$ This special case can be demonstrated by/to students using beads/keys on a necklace/ring.

Examples/Activities

Some exercises should be assigned for this concept.

1. How many ways can we arrange five objects in a line, if the only difference is colour, and there are two red, one green, and two yellow?

2. In how many ways can we arrange the letters of the word 'defense'?

3. In how many ways can we arrange the letters of the word 'inference'?

4. How many different signals can be formed from three squares, three triangles, and two circles, if all eight symbols must be displayed in a vertical line?

Adaptations

These exercises are meant to be completed using the formula in the resource texts. It is a variation of

$$\frac{n!}{n_1!n_2!n_3!\ldots}$$

where n_1, n_2, n_3 ... represent the number of times an element appears in the arrangement.

This topic can be extended by introducing many other real-world applications. Students could be asked to generate these in groups, and set up related questions to solve. (IL)

Examples/Activities

1. Determine the number of permutations of four (five, six) objects arranged in a circle.

2. In how many ways can a family of five be seated at a circular table?

3. In how many ways can six children be seated on a small merry-go-round that has exactly six places?

4. In how many ways can we arrange five keys on a key-ring?

5. In how many ways may we seat a family of six at a table, if Bob and Joan are to sit next to each other? If Bob and Joan are not to sit next to each other?

6. In how many ways can seven go-karts be situated on a circular track during a race, assuming that no two are exactly side by side?

Adaptations

The questions given to the students can be adapted quite readily for better students by introducing restrictions similar to number 5 in the examples/activities column.

For example, students could attempt questions such as:
In how many ways may we seat three couples (men and women) at a table, if men and women must be seated alternately?

In how many ways can we seat five people in a circle, if we do not want Bill and Bernie to sit next to each other?

In how many different ways may we arrange seven objects in a circle, if two of the objects are identical?

Concept A: Permutations and Combinations

Objectives

A.6
To determine the number of combinations of n objects, taken r at a time.

Instructional Notes

Students should be presented with the definition of combination, and the major differences of combinations and permutations should be pointed out; that for combinations, the order of arrangement is not important.

Groups could work on a set of exercises designed to build an understanding of the underlying principles of combinations.

These exercises should be chosen to reflect or model real-world situations, and manipulatives should be available to the groups to use in their explorations. (NUM)

Examples/Activities

Some types of examples are:

1. How many ways can we choose one representative from a group of three? four? five?

2. How many ways can we choose two committee members from a group of four? five? ten?

3. How many possible combinations are there in Lotto 6/49?

4. How many line segments can be drawn using a set of twelve non-collinear points? How many triangles? How many quadrilaterals?

Adaptations

This topic could also be studied by using a queue of length r taken from n people. Pick r from n people and then line them up in r! different ways. Then we have
${}_nP_r = {}_nC_r\ r!$ and therefore,

$${}_nC_r = \frac{n!}{(n-r)!\ r!}$$

Note that the two above methods are equivalent for repetitions; for example, the number of permutations of the letters of MISSISSIPPI is

$$\frac{11!}{4!\ 4!\ 2!\ 1!}$$

which can be seen as ${}_{11}C_4 \cdot {}_7C_4 \cdot {}_3C_2$ by picking places for each set of letters and using the multiplication principle. Try others to determine if this equivalence is general for repetitions.

Objectives

A.7
To determine the number of combinations formed from more than one subset.

Instructional Notes

Students should be reminded of the fundamental counting principles before beginning this topic.

Groups could work on the solutions of example problems that reflect this situation. Group answers, and the rationale for these answers, should be shared with the class. (COM)

The examples chosen should try to reflect real-world situations insofar as possible.

An introductory problem might be posed to the students to work on cooperatively. An example might be similar to the following:

An art student is instructed to make a collage of 2 different triangles, 3 different quadrilaterals, and two pentagons. In the supply material provided, there are six different triangles, 5 different quadrilaterals, and 4 different pentagons. In how many different ways may the student choose the materials required?

Examples/Activities

Examples:

1. How many different committees are possible, if we would like to have three men and three women on the committee, and the number of men to choose from is 5, and the number of women to choose from is 6? (Manipulatives e.g.; using 5 green markers for men, and 6 blue markers for women, could be used in a simulation - for group work.)

2. In a group of scientists selected for astronaut training, there are 15 men and 4 women. Of this group, only four are to be chosen for the actual training program. It is advisable that there be exactly one woman included. How many different selections can be made?

   What would the result be if two women must be chosen? What is the result if at least one woman must be chosen? What are some possible reasons that the target group seems to be skewed towards more males in this scenario? How might it be made more accessible to both sexes?

3. To put together an investment portfolio for a client, a stockbroker has suggested to a client that she choose four of the six recommended stocks, and three of the eight recommended bonds. How many different selections can the client choose?

Adaptations

The groups can extend their skills and knowledge of this topic by researching examples of combinations present in other real-world situations; environmental, hereditary, games, politics (such as the constitutional debate over Senate reform), economics, and others. These could be done as simply listing situations where combinations occur, or mini-reports could be expected reflecting writing skills and some computations as well.

Students could work on a specific situation such as determining the number of ways it is possible to choose a dozen doughnuts from five varieties, if at least one of each variety must be chosen? Two of each?

Student discussion on this type of problem may lead to other concepts (such as the binomial theorem) which could be briefly discussed at this time, if student interest warrants.

Concept B: Data Analysis

Foundational Objectives

- To demonstrate developed skills and understanding in collecting and displaying a set of data for a given situation (10 02 01). Supported by learning objectives 1 to 4.
- To provide reasonable explanations of the interpretation of a set of data (10 02 02). Supported by learning objectives 5 and 6.

Objectives

B.1
To list and describe the methods used to collect data.

Instructional Notes

Students, in small groups, pairs, individually, or as a class, should be asked to generate a list of different ways in which data is collected. (This should be a review for them of concepts done at the Middle Level.) Once the list is felt to be comprehensive enough, the teacher can ask students working in groups to describe each of the listed methods in more detail, and to provide an example or two where each method could be used.

The students should be asked also about the limitations of each method, and to discuss the ways in which it is possible to obtain skewed results from using various methods. (CCT)

The discussion should determine also whether each method confines itself to the specific target group, whether it misses certain segments of the target group, or whether it might be totally invalid because of missing most of the target group.

The teacher should have specific examples in store, in case the class does not generate sufficient examples to illustrate these methods.

Examples/Activities

Students should discuss the data collected and displayed for situations such as the following:

1. The average height of teenagers in Canada, after measuring high school boys basketball teams playing in BRIT, a tournament at Bedford Road Collegiate in Saskatoon.
2. The advisability of stocking rock music videos in a music store, if the store caters to symphony members only.
3. The suitability of women for executive positions, by asking only executives who happen to be men.
4. Using the phone to conduct a survey, when results are to reflect all opinions.
5. Graphs showing growth or decline curves representing a company's point of view. (Scales should be inspected and alternative scales suggested, to give a different perspective.)

Adaptations

Students could be asked to do some research in the learning resource centre to locate information on a point of view supported by some data (e.g.: environmental (mis)management, unemployment figures, memberships and purchase patterns in a local co-op, various polls or surveys, topical issues, makeup of consumer price index, etc). Students should be encouraged to access electronic databases for the information. (TL)

They could be asked to analyze the methods in which these data were gathered and presented, and put forward their opinion on whether this particular situation was presented in a fair manner. Reasons should be given for their opinion. (CCT)

Objectives

B.2
To obtain data for real-world situations by using simulations (such as Monte Carlo simulations).

Instructional Notes

Students should discuss how they would collect data in specific situations. They should be presented with situations in which they realize that the data necessary cannot be collected directly, but must be obtained by other means. The concept of a simulation should be introduced as a mathematical tool, as the simulation of driving skills in a simulator is a tool for driver education. (TL)

Once the students are introduced to the concept of simulations, they can discuss how simulations can be used in mathematics. Most of these simulations will require manipulatives such as coins, various types of dice, cards, or spinners. Random number tables can also be useful.

Group work will enable the students to complete many more trials of a particular situation in less time, as well as have these trials recorded.

Time must be taken to outline the procedures to be followed.

Examples/Activities

Data collection by simulation requires certain types of exercises. Some distributors of educational materials have resources available for purchase by schools.

Some examples of simulations follow.

1. Krunchies breakfast cereal for kids places 1 of 6 possible figurines in each box as a premium for kids. (Each of the six figurines has an equal chance of being placed in each box.) On average, how many boxes of Krunchies would Kara have to purchase in order to obtain all six figurines?

   Solution: The simulation could involve the repeated roll of a die, where each of the numbers 1 through 6 would represent a figurine. One trial would be the instance of rolling the die until all six numbers had appeared at least once. (How many times was the die rolled to get the six numbers?) Each group could run a specified number of trials, or trials for a specified time, and the results of the class would be collated, with the final calculations being done together.

2. If Ali S. makes 75% of her free throws, is there a reasonable expectation that she will make two free throws in a row?

   Solution: A spinner can be used, where three fourths of the spinner indicates a successful free throw. (Random number tables, dice, or cards can also be used.)

Adaptations

To extend this topic, the question could be reworded to have one or more of the figurines limited in its random placement e.g.: Figurine A is included one third as often as the other figurines. The students would then have to allow for this in setting up the trials in their simulation.

Many other simulations could be done, involving genetics from biology, politics from social studies, and environmental concerns from other areas of science. (IL)

There are a number of resource packages available, most of which are described in the catalogues of educational publishers.

Objectives

B.3
To review the methods for determining the measures of central tendency.

Instructional Notes

Students should have some previous knowledge of the terms mean, median, and mode from their mathematics in Middle Level, but a review of these terms is necessary at this stage.

Examples that describe and illustrate how these measures of central tendency are determined should be presented. Students might be asked to discuss the suitability of a particular measure in a given situation and to determine which measure might be most suitable for that particular situation.

Examples/Activities

Find the three measures of central tendency in each of the following situations, and determine the measure(s) that are most suitable for the situation described, supplying your rationale for your choice(s).

1. The salaries received by the starting five of the Northwood Rebounders basketball team are $240 000, $175 000, $225 000, $200 000, and $2 300 000.

2. A T-shirt company finds that its sales for one month are as follows;

S	-	125 @ $ 8.00
M	-	350 @ $ 8.50
L	-	775 @ $ 9.00
XL	-	550 @ $ 9.50
2XL	-	125 @ $10.00

3. The test marks for a unit test in mathematics are:

67	86	92	58	77	81	45
73	74	59	67	75	36	99
72	63	70	84	68	71	78
61	52	82	77	90	77	83
66	73	77	83	65	79	57

4. The results of the high jump event (in metres) at a recent track meet are as follows:
   2.04, 2.01, 1.95, 1.89, 1.89, 1.83,
   1.80, 1.75, 1.75, 1.70, 1.65, 1.65,
   1.65, 1.60, 1.60, 1.60, 1.60, 1.60

Adaptations

To modify, the teacher may ask students to state their shoe size, amount in cents of the coins they have on their person, or the number of pens or pencils they have with them in class. (If students are sensitive to such ideas as shoe sizes, etc., do not press for a response.) The class can then work in small groups with the generated data to determine the three measures of central tendency, and discuss which are the most useful in each case, why, and for whom.

The more able students might be given a more detailed research assignment based on these measures. They might be asked to contact the municipal office to determine average property tax in their municipality, the median tax, and the mode. They might be given an assignment based on the amounts of various chemicals in a selection of dry cereals. This could be noted from the packages in a local grocery. (IL)

Concept B: Data Analysis

Objectives

B.4
To construct box and whisker plots from simulated data.

Instructional Notes

In order to begin this section, students should have a basic understanding of median, and the definition of quartiles. These are necessary in order to set up box and whisker plots.

Students should have some experience in calculating the first and third quartiles before constructing these plots.

Definitions of terms such as 'outliers' or 'extreme values' should be introduced.

Many of the newer textbooks and resource materials will introduce this topic in a basic format and will provide examples to use.

Examples/Activities

Construct box and whisker plots for the following sets of data.

1. The test marks from a unit in mathematics were:

62	77	83	74	67	90
38	85	75	66	88	57
59	65	81	74	98	86
62	72	81	79	65	83

2. David's scores in basketball league games this past winter were:

12	10	9	14	18	14	13
11	25	15	16	12	8	2
18	21	14	11			

3. The number of boxes of chocolates sold in each area of a city in a recent fundraising activity were:

212	395	264	285	207
296	259	103	238	249
253	274			

Adaptations

Students could be asked to formulate plausible explanations for the outliers in each of the example questions and to try to determine what effect the outlying values have (or could have) on the set of data described.

Students could then be asked to locate a set of data pertaining to a real-world situation (e.g; political, social, environmental, geographical), construct a box and whisker plot, and then analyze it in a similar fashion. (IL)

Objectives	Instructional Notes
B.5 To define and utilize the concept of percentiles (including the first, second, and third quartiles).	The definition of percentile should be introduced, and some examples given of the use of percentiles. After completing this section, students should be expected to give a reasonable description of statements involving percentiles. Students could work in groups to deal with data they have supplied in this section. This topic is not meant to be covered in depth.

Examples/Activities

Give your interpretations of each of the following statements.

1. When having a physical examination, Oscar was told his height was at the 90th percentile for his age.
2. In the preceding example, Oscar's weight was at the 60th percentile for his age.
3. Describe Oscar's physical build in general terms, based on statements 1 and 2.
4. In comparing her net income to a table of incomes for all workers obtained from Stats Canada, Victoria found that her income was at the 40th percentile. In comparing her income to the table for incomes of female workers, the same income placed her at the 55th percentile. (E-stat CD-ROM from Statistics Canada is a good source.)

Adaptations

To extend this topic, students could be asked as a class to obtain data, such as nutritional information on grams of fat, sugar, etc in a serving of dried cereal, order this information, and determine which cereals are:

i) at or above the 60th percentile in terms of fat content,
ii) above the 90th percentile, and,
iii) below the 20th percentile. Then, the students should make recommendations as to which types of cereal would be best for low-fat diets. (Low fat cereals should be checked for other ingredients as sugar or salt.) The lowest fat content may not necessarily be the recommendation expected.

Objectives

B.6
To solve related problems using statistical inference.

Instructional Notes

Students should be given sets of data to analyze in groups. The methods developed in this unit should be utilized. The expectation for this objective is that the students will analyze the data and be able to present reasoned explanations for their conclusions.

The answers expected should not consist only of the students' conclusion, but their calculations and a written statement of their reasoned conclusion. Students could discuss the conclusions reached by various groups, to try to understand how others reached their conclusions as well. (CCT)

Examples/Activities

Examples that could be used might be similar to the following:

1. Give the students information on the number of goals scored by Maurice "The Rocket" Richard, Gordie Howe, Bobby Hull, Wayne Gretzky, and Brett Hull. (These can be obtained from hockey cards, or sports record books.) Have the students note the number of games played by each, assists by each, and ask them to determine who is the best goal scorer in this group.

   Each group should present its reasoned answer. There may be more than one acceptable answer, as some may use totals only, others may use scoring average per game, some may use only goals, and some may use arguments based on the opposition that the players faced at the time they played. Statistics for any group of "greats" in any sport or activity could be substituted.

2. Take the results of any recent Gallup, or Decima poll, with information about how the poll was conducted, and ask the students to analyze the results stated. What are the variables that might affect the poll?

Adaptations

Students could be asked to locate data on a given topic, analyze the data they obtain, and present some logical conclusions based on this data. They should also be expected to outline the limitations of their analysis. (CCT)

Concept C: Polynomials and Rational Expressions

Foundational Objectives

- To demonstrate ability in the addition, subtraction, multiplication, and division of rational expressions (10 03 01). Supported by learning objectives 1 to 7.
- To demonstrate ability in solving equations involving rational expressions (10 03 02). Supported by learning objectives 8 and 9.

Objectives	Instructional Notes
C.1 To factor the difference of squares of special polynomials.	Although students have already been exposed to factoring the difference of squares in previous grades, a brief review of the technique may be necessary. The review may also encompass other areas of binomial and trinomial factoring, emphasizing common factors, grouping, and trinomial squares. Once the introductory review is complete, the factoring of special polynomials can be introduced. This type of factoring usually requires a two-step approach and students should obtain some practice in identifying the steps necessary. When students are able to visualize the needed steps, the factoring can take place. E.g.: In order to factor $x^2+2xy+y^2-m^2$, students should be able to visualize the question as $(x^2+2xy+y^2)- m^2$, where $x^2+2xy+y^2$ must be factored first as a trinomial square to obtain $(x+y)^2$, and then $(x+y)^2-m^2$ factored as a difference of squares.

Examples/Activities

The students might work at a set of exercises similar to the following:

1. $x^2+6x+9-25$
2. $y^2-4y+4-9n^2$
3. $x^2+8x+16-4m^2$
4. $x^2-6x+9-(x-4)^2$
5. $2y^2-20y+50-8z^2$
6. $(3x+2y)^2-(x-3y)^2$
7. $(x^2+4xy+4y^2)-(9x^2-6xy+y^2)$
8. Students could perform some mental math using this concept.
   a) 33 x 27 =
   b) 62 x 58 =
   c) 23 x 27 =*
   d) 31 x 39 =*
9. Translate into mathematics:
   " Take a number, add three, square the result, and subtract twenty-five."

* To square numbers ending with the digit 5, simply multiply the first digit (n) by the next consecutive digit (n + 1), and place 25 after this product.
Therefore 23 x 27
= (25 - 2) (25 + 2)
= $25^2 - 2^2$
= (2 x 3)25-4
= 625 - 4
= 621

Adaptations

Better students could be instructed to find a real-world application of this concept by searching through various resource books the teacher may have on hand. When such examples are found, it might be worthwhile to discuss whether they reflect actual situations, or are contrived to fit the concept.

This might be illustrated to students using manipulatives, showing that $x^2 - y^2$ is equivalent to $(x + y)(x - y)$.

An alternative question might be to ask students to multiply any three consecutive numbers and then add the product to the middle number. What is the result? Ask them to attempt to show this is true for all cases.

As an example, 6 x 7 x 8 = 336. Add 7, which is the middle number, for a total of 343. Note that this is 7^3.

For the general case, one way of proving this is to note that $(n-1)(n)(n + 1)$ is the product of the consecutive numbers. This multiplies to $n^3 - n$. Adding the middle number, n, gives a total of n^3.

Better students can relate this to the difference of squares method. (CCT)

Concept C: Polynomials and Rational Expressions

Objectives	Instructional Notes
C.2 To factor the sum and difference of cubes.	This topic can be introduced in several ways. It can be treated as a formula, done by long division, or three-dimensional cubes can be used as manipulatives to help students visualize what it means to factor a cube. The method(s) chosen will depend on time available, and the needs of the students in the class. The outcome is that the student will be able to factor these cubes by inspection.
C.3 To factor polynomials using the factor theorem.	A review of the definition of a factor, and of evaluation by substitution, could be useful in introducing the factor theorem to the students. FACTOR THEOREM: (x-a) is a factor of f(x) if and only if f(a) = 0. The correlation of x-a to f(a) must be highlighted. There are slightly different versions of the factor theorem, as presented in various series of textbooks, and you may wish to utilize your personal favourite. Students should be given some questions to determine if a binomial is a factor of a given polynomial. This will allow them to gain the required practice of how to determine a factor before proceeding to the next step of determining the other factors of the polynomial.

Examples/Activities

Students could work individually, in small groups, or as a class, to factor some examples of the following types:

1. x^3+27
2. x^3-y^3
3. $27x^3+y^3$
4. $8x^3-125y^3$
5. $64x^6+27y^9$
6. $x^9y^{12}-z^{15}$
7. $81x^3+192y^6$
8. $(a+b)^3-216$
9. $(x-2)^3+27$
10. $(2x+3y)^3+(x-2y)^3$

The introductory exercises would require the students to determine if the given binomial is a factor of the given polynomial. (Note that **not all** examples should be factors, although **some should** be.)

1. Is (x-4) a factor of $3x^2-7x-20$?
2. Is (x+3) a factor of $2x^3+7x^2+x-6$?
3. Is (x-3) a factor of $4x^3-5x^2+8x-15$?
4. What are some possible factors of $x^3 + 3x^2 - 10x - 24$? How did you arrive at this conclusion? Test your answers. Can you state a generalization based on your answer? (This type of question might be used as an introduction to this concept, using an inquiry approach.)

The second set of exercises should ask students to determine what the factors of the polynomial are, if f(a) = 0.

1. Repeat the first set of questions and determine all factors of the polynomial, where the first binomial was determined to be a factor of the polynomial.

Adaptations

Students could be asked to do research using mathematics texts to locate real-world problems that involve factoring of the sum and difference of cubes.

A discussion of any such problems could focus on the types of applications, who would need to do these types of questions, and whether these problems are realistic or contrived. (COM)

Students might be asked to factor these by using computers or calculators. (TL)

Students could be introduced to synthetic division to complete the factoring of a polynomial.

The evaluation by substitution could be done on a calculator or computer.

For further exploration of this topic, students could be asked to determine the effect of a constant 'a' being included in the binomial (ax + b). Discuss how this constant could be accommodated in the general pattern of factoring polynomials.

E.g.: is (2x-1) a factor of $6x^3-5x^2+3x-2$?

E.g.: is (3x+2) a factor of $12x^2-7x-10$?

Students could also be asked to determine a missing coefficient(s) or value(s), as in the following examples;

a) What value does c have to be, if (x - 3) is a factor of $3x^2 - cx + 3$?

b) What is the value(s) of b, if any, if (x - 2) is a factor of $2x^3 + x^2 - bx + 6$?

These types of questions could be extended to include two unknowns for more able students.

Objectives	Instructional Notes
C.4 To use the remainder theorem to determine the remainder when a polynomial is divided by (x-r).	In general, the remainder theorem may be stated as; when f(x) is divided by x-r, the remainder is f(r). For the example, given $(x+3)\overline{)x^2+10x-2}$ students might be asked to provide the division statement (x+3)(x+7) -23, or they might be asked to determine the quotient and remainder, e.g.: quotient (x+7), remainder -23, or as x+7 $\frac{-23}{(x+3)}$ Students should practise determining the remainder mentally, using evaluation by substitution of (x-r) into f(x) to calculate f(r). Less able students might do this evaluation using a calculator. (TL)
C.5 To simplify rational expressions involving opposites.	This topic is an extension of simplifying as done in Mathematics 20. The objective is to introduce the students to the property a/-a = -a/a = -1, and to be able to incorporate this property into their work with rational expressions. Some review of simplifying expressions may be needed to begin this topic. The teacher may wish to have the students 'discover' this rule by themselves by working through several examples and generalizing from the examples.

Examples/Activities

A set of exercises such as the following might be suggested;

1. For $f(x) = 3x^3-4x^2+7x-4$, determine the value of: f(3), f(-2), f(2), f(4). For each of these, write the division statement.

   Example. Since f(3) represents the binomial (x - 3), the division statement could be written $(3x^3 -4x^2 + 7x -4) \div (x-3)$.

2. For $f(x) = 3x^2+5x-6$, write the division statements for f(-5), f(2), and f(4).

3. Find the quotient and remainder for each of the following:
   a)
   $(x-4)\overline{)3x^2-x+8}$

   b)
   $(x+5)\overline{)4x^2-3x^2+7x-1}$

   c)
   $(x-2)\overline{)2x^4-3x^2+6}$

Students could work individually to simplify the following expressions, or could simplify and discuss the solutions in small groups.

1. Simplify each of the following: (Also list any non-permissible values for the variable.)

   a) 6/-6 d) $-5x^3/5x^3$

   b) -21/21 e) -(2x+3)/(2x+3)

   c) 3x/-3x

2. Use factoring to simplify each of the following, noting any non-permissible values for the variable.

   a) (4x-10)/(-2x+5) c) $(3x^2-10x+3)/(1-3x)$

   b) $(x^2-4x-21)/(7-x)$ d) $(16-9x^2)/(x-4)$

Adaptations

Similar to the last concept, this topic may be extended by having students determine the value(s) of missing coefficients.

Example.

Determine the value(s) of m when $(2x^3 - x^2 - mx + 21) \div (x - 2)$ has a remainder of 3.

This topic could be extended by introducing exercises where the students would have to factor both the numerator and denominator before simplifying.
E.g.: simplify
$(6x^2-x-12)/(-8x^2+2x+15)$

Objectives	Instructional Notes
C.6 To add and subtract rational expressions with polynomial denominators.	Students have taken addition and subtraction of rational expressions in Mathematics 20 but only with monomial denominators. This section extends the concept to polynomial denominators. A brief review of the topic may be necessary. Students should be able to work in small groups to simplify and discuss exercises in this section. (PSVS)
C.7 To multiply and divide rational expressions involving opposites.	A brief review of the general technique required to simplify these expressions may be necessary. The examples utilized for review may be numerical, algebraic, or a combination of both. Students should be expected to identify non-permissible values where these exist. If working in small groups, students may apportion each question so that each group member gets practice factoring polynomials. E.g.: student A might factor the first numerator, student B the first denominator, student C the second numerator, and so on. In this way, each contributes to a group effort, and each receives individual practice in factoring. Students can check each others' factors by multiplication. (PSVS)

Examples/Activities

Students should be given a series of exercises which provide practice in adding or subtracting rational expressions. They should also be instructed to note any non-permissible values for variables.

1. $3/5 + 2/3$
2. $4/5 + 1/4 - 5/6$
3. $5/3x + 4/2x$
4. $2/5x - 3/4y$
5. $6/(2x-3) + 5/(3-2x)$
6. $4/(x-2)+ 3/(x+1)$
7. $(7x-1)/(x+3) - (2x+1)/(x-1)$
8. $3/(x^2-x-12) + 2/(x^2-16)$
9. $(2x-5)/(x^2+3x-40) - (3x+2)/(x^2-x-72)$
10. If time is equal to the distance divided by the rate, write a mathematical expression for the total time taken by both cars in the following:

    Car A travels 120 km at a certain speed, while Car B travels 150 km at a speed 10 km/hour faster than Car A. Simplify.

Examples or exercises similar to the following might be chosen:

Simplify each of the following.

1. $(6/-5) \bullet (13/10)$
2. $(7/3x) \bullet (5x^2/-56)$
3. $\frac{2x+10}{14-2x} \bullet \frac{x-7}{x+5}$
4. $\frac{x^2-5x-14}{4-x^2} \bullet \frac{x^2-5x+6}{3-x}$
5. $\frac{2x^2-7x+3}{6-x-x^2} \div \frac{4x^2+5x-6}{15-7x-4x^2}$

Adaptations

Students might be instructed to formulate the procedure that could be used for adding and subtracting all types of rational expressions. This might be done in the form of a written algorithm, an algebraic exercise, a written paragraph, an oral presentation (with illustrations), programming a calculator or computer, or a series of completed examples, depending on the capability of the student(s). (COM)

This topic can be extended by including exercises that contain three or more rational expressions, and also include combinations of multiplication and division. Parentheses may also be included to provide further alternatives.

E.g.:

$$\frac{25-4x^2}{1-9x^2} \bullet \frac{3x^2+10x-8}{4x^2-4x-15} \div \frac{6x^2+7x-3}{2x^2+13x+20}$$

Objectives

C.8
To solve and verify linear equations in one variable involving rational algebraic expressions (including polynomial denominators).

Instructional Notes

Students could be given an assortment of simple linear equations of the types they have done in previous grades. They could be instructed to outline the basic procedures in solving equations, e.g.: simplify both sides, isolate the variable, solve for the variable, and check the solution.

Once they have reviewed the strategies involved, they could practise by solving the review equations, individually, in pairs, in small groups, or as an entire class.

When the review has been completed, they may be introduced to equations involving rational algebraic expressions.

They should be asked to determine if the basic procedures remain the same, or if additional steps are required. If so, what steps?

Students can work in pairs or in small groups to solve several of these equations.

Examples/Activities

A typical student set of this type of equation might include equations similar to the following:

1.

$$\frac{4}{3x} + \frac{2}{5x} = 13$$

2.

$$\frac{3}{x-1} = \frac{2}{x+1} + 1$$

3.

$$\frac{4}{x+2} - \frac{1}{x-2} = \frac{5}{x+2}$$

4. Translate the following statements into mathematical equations, and then solve for the variable(s).
   a) The time taken to drive 150 km at a certain speed is exactly the same as the time it takes to drive 120 km at a speed which is 15 km/hour less.

   b) The lengths of two different rectangles are the same. The first rectangle has an area of 2 400 cm^2, and the area of the second rectangle is two thirds as much. The width of the first rectangle is 20 cm more than the width of the second.

Adaptations

Students could be asked to locate any instances where these types of equations are used in real-world applications. If any are found by the students, they could be shared with the class. If students are not successful, the teacher might provide one or two instances where this type of equation is used. (IL)

The students could then be instructed to solve the equations generated by the real-world examples.

As an alternative, the teacher may begin by having the students solve some stated problems on their own, and then formalizing the procedure using their results and solutions. (CCT)

Objectives

C.9
To solve and verify the solutions of quadratic equations involving rational algebraic expressions.

Instructional Notes

Students should do a brief review of the techniques used in solving quadratic equations, and be given a few equations to solve. (At this stage, they have solved quadratic equations only by factoring and by taking the square root of both sides of the equation.)

Once the review is complete, students can be given a few equations with rational algebraic expressions to solve. These can be done as a class, individually, in pairs, or in small groups. The importance of checking solutions should again be emphasized.

Examples/Activities

Students can be instructed to solve equations similar to the following:

1.
$$\frac{5}{x-3} = 1 + \frac{30}{x^2-9}$$

2.
$$2 + \frac{6}{x-1} = \frac{12}{x^2-1}$$

3.
$$\frac{1-5x}{x+3} = \frac{3x+1}{x^2-9} + \frac{x+3}{3-x}$$

Some word problems similar to the following can be given to the class to work on in small groups.

Translate the following statements into a mathematical equation involving rational algebraic expressions. Solve for the variable and verify the solution.

Joan drives 200 km at x km/hour. Franco then takes over, and drives 200 km at (x - 10) km/hour. The total time taken by the drivers is 9 hours. How fast did each drive?

Adaptations

Students could be asked to search through reference books to find examples where these types of equations are used in real-world situations. Reference books from other areas, such as commerce, administration, engineering, medical applications, and others could be utilized, as well as various mathematical texts. (IL)

Any examples found can be shared with the entire class and solutions to the examples can be attempted.

Word problems involving these types of equations should also be presented to the class.

Concept D: Exponents and Radicals

Foundational Objectives

- To be able to illustrate the relationship between the radical and exponential forms of an equation (10 04 01). Supported by learning objectives 1, 3, and 5.
- To demonstrate the ability to work with operations involving radical numbers (10 04 02). Supported by learning objectives 2, 4, 6, and 7.
- To be able to solve equations involving radicals and to be able to justify the solutions (10 04 03). Supported by learning objectives 8 to 11.

Objectives

D.1
To evaluate powers with rational exponents.

Instructional Notes

Students should be asked to recall the basic laws of exponents and to demonstrate their understanding of integral exponents. This could be done as a class activity. These could be listed as they are completed, so all students will have the opportunity to recall the basic facts.

The students may be asked to consider the likely values of an expression of the type $25^{1/2}$. Students could discuss possible answers in small groups and be instructed to provide a rationale for their group answer.

If no group seems to approach the solution, the teacher could provide a series of clues for the students to utilize, beginning with the properties of exponents.
E.g.: $25^{1/2} = (\ ?\ )^{1/2} = (5^2)^{1/2} = (5)^{2(1/2)}$

$= 5^1 = 5$

In the above example, the students should develop, or be shown, the relationship of an exponent of 1/2 and the square root. The notation for a rational exponent can be introduced,

$(x^{1/m} = \sqrt[m]{x^1})$,
and several examples done by the teacher, class, or individual students. The definition can be extended to

$$X^{n/m} = \sqrt[m]{X^n}$$

and more examples considered.

Examples/Activities

A few exercises in evaluation of powers with rational exponents could be given to students to check their understanding.

1. $49^{1/2}$
2. $27^{1/3}$
3. $8^{2/3}$
4. $32^{3/5}$
5. $4^{-1/2}$
6. $64^{-2/3}$
7. $(-125)^{2/3}$
8. What would happen in the example $(-16)^{1/4}$? Should we qualify our definition? How would we redefine it to cover all possibilities? How does your math text handle this? How do other math texts deal with this situation?

Adaptations

On an intuitive level, students could be asked to determine the length of a side of a cube whose volume is $343\ cm^3$, $x\ cm^3$, $(x - 3)^3\ cm^3$, or $(x^3 + 12x^2 + 48x + 64)\ cm^3$.

Students' understanding could be checked by asking related questions such as the following;

a) Which is the larger of $8^{2/3}$ and $25^{1/2}$?

b) Arrange in order from smallest to largest:

$64^{2/3}$, $64^{1/2}$, $64^{-1/3}$, $\left(\frac{1}{64}\right)^{-1/2}$

c) If the volume of a sphere is given by the formula $V = (4/3)\pi r^3$, and the volume is known to be $(153/6)\pi\ cm^3$, find the radius r.

Concept D: Exponents and Radicals

Objectives	Instructional Notes
D.2 To apply the laws of exponents to simplify expressions involving rational exponents.	Some statements could be given to the students to work on in pairs or in small groups. These should give the students some practice in working with rational exponents. E.g: a) $x^{2/3} \bullet x^{5/3}$ b) $(x^{1/2})^{2/3}$ Once the students have had practice in working with the laws of exponents, this can be extended to simplification, or evaluation exercises as well. E.g.: $x^{1/2} \bullet x^{1/3}$ $8^{1/2} \bullet 8^{1/6}$
D.3 To write exponential expressions in radical form.	Students will be expected to be able to demonstrate that they understand the definition of a rational exponent by writing a given radical expression in radical form. Students should be instructed to review their definition of a rational exponent and to apply that definition to a set of exercises in which they would practise this concept. They may work individually, in pairs, small groups, or as an entire class, in practising this concept. (PSVS)

Examples/Activities

A sample set of exercises might include examples similar to the following:

1. $x^{2/3} . x^{1/4}$
2. $(x^{1/3})^6$
3. $(x^{3/4} . y^{2/3})^{6/7}$
4. $x^{3/5} y^{1/3} . x^{-5/6} y^{3/4}$
5. $\frac{x^{7/5}}{x^{5/6}}$
6. $8^{2/3} . 16^{3/4}$
7. $25^{-1/2} . 1000^{2/3}$

Rewrite each of the following exponential expressions in radical form.

1. $x^{5/3}$
2. $x^{1/4} y^{3/4}$
3. $(x^3y^5)^{1/4}$
4. $27^{4/3}$
5. $16^{3/4}$

Adaptations

The teacher may wish to assign questions that compare two expressions that involve rational exponents. Example: Which of the following expressions is greater than the other?
$x^{2/5} . x^{3/2}$ or $(x^{1/3})^4 . x^{5/6}$

Is this true for all values of x? Can we find values of x where each is greater? Use your calculator to determine these results.

Objectives

Instructional Notes

D.4
To simplify square root and cube root expressions.

A review of the simplifying process of square roots as studied in Mathematics 20 may be a useful starting point.

When the students have recalled the principles of simplifying square root radicals, they can be introduced to cube root simplification.

It may be necessary to have the students compare the square root and the cube root of various numbers, to help them discover that the cube root of a negative number does in fact exist. This can be tied to the definition of a rational exponent and aid understanding of the role of an even or odd denominator in rational exponents.

There are several different methods by which this concept can be explained; check your reference texts for different explanations. Some students may prefer to look at some of these options.

Note that one common error that students tend to make with √ is

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}$$

Try some numerical examples to show this is not the case.

Ask students if there are any cases where

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}?$$

D.5
To write radical expressions in exponential form.

Students are expected to become familiar with converting from exponential expressions to radical form and vice versa. In objective D.3, the former was introduced and practice given.

The teacher may find it useful to combine objectives D.3 and D.5 in one lesson.

In any case, some practice should be given to the students to work on converting from radical to exponential form. The students could practise individually, in pairs, or in small groups.

Examples/Activities

The exercises should provide some numerical and variable expressions. Simplify.

1. $\sqrt{27}$

2. $\sqrt[3]{16}$

3. $\sqrt{18x^5}$

4. $\sqrt[3]{54x^2y^5}$

5. $\sqrt[3]{343\,x^8y^7z^6}$

A set of exercises might include examples similar to the following:

Rewrite in exponential form.

1. $\sqrt{x^5}$

2. $\sqrt[3]{x^4y^5}$

3. $\sqrt[5]{2^6x^8y^9}$

4. $\sqrt{3^3}\ \sqrt[3]{3^2}$

Adaptations

Students could also be expected to attempt some written questions similar to:

For what value(s) of the variable is $\sqrt[5]{64\,x^6}$ greater than $\sqrt[4]{16x^4}$?

Objectives

D.6
To add, subtract, multiply and divide square root and cube root expressions.

Instructional Notes

Square root operations were introduced in Mathematics 20. Students may be asked to recall the basic procedures involved in these operations. Students can be given a few exercises to do to review these Mathematics 20 operations.

The introduction of cube root operations can be done as an entire class or by getting individual or group input when developing these procedures.

Students could be given a set of exercises to use in developing their skills in dealing with square and cube root operations. Students could work in pairs or in small groups in doing these exercises. (PSVS)

Examples/Activities

A variety of exercises involving all four basic operations could be used. A combination of operations could also be used once students have become familiar with the basic operations.

1. $3\sqrt{8} + 4\sqrt{18} - 2\sqrt{32}$
2. $6x\sqrt{12y} - 2\sqrt{27x^2y} + 4x\sqrt{3y}$
3. $2\sqrt[3]{16} + 4\sqrt[3]{250} - 3\sqrt[3]{54}$
4. $x\sqrt[3]{xy^5z^4} + 2yz\sqrt[3]{x^4y^2z}$
5. $4\sqrt{3}\,(2\sqrt{6} - 5\sqrt{15})$
6. $(4\sqrt{5} + 2\sqrt{3})(3\sqrt{5} - 7\sqrt{3})$
7. $(8\sqrt{3} - 2\sqrt{6})^2$
8. $4\sqrt[3]{5}\,(3\sqrt[3]{50} + 7\sqrt[3]{75})$
9. $\dfrac{15\sqrt{12} - 8\sqrt{24}}{6\sqrt{3}}$
10. $\dfrac{4\sqrt[3]{18x^2y^4} + 6\sqrt[3]{12x^4y}}{2\sqrt[3]{6xy}}$

Adaptations

Better students should be encouraged to attempt exercises where the indices are not the same for all terms.
E.g.:

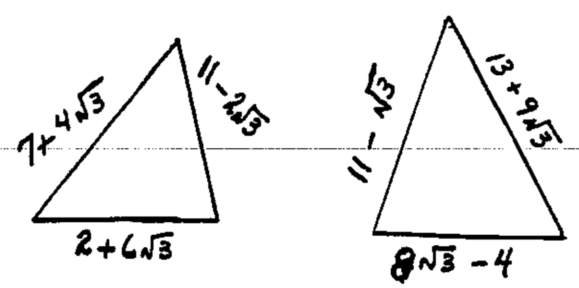
$$2\sqrt{6} \bullet 3\sqrt[3]{6}$$

As well, students could be given exercises where the indices are numbers other than 2 or 3.
E.g.:

$$2\sqrt[5]{x^6y^7} \bullet 3\sqrt[5]{x^7y^9}$$

Students should also work with word problems similar to the following:

1. Which of the triangles below has the greatest perimeter?



2. Square A has sides of length $7\sqrt{2} + 6\sqrt{3}$, while Square B has sides of length $4\sqrt{5} + 3\sqrt{3}$. Which square has the greater area?

Concept D: Exponents and Radicals

Objectives

D.7
To rationalize monomial and binomial denominators in radical expressions.

Note: Objectives 1 through 7 are generally covered in most textbooks. The order of the objectives may be somewhat different than is listed here. The teacher should endeavour to make sure that all the objectives are covered, but the order of presentation will depend upon the preference of the teacher, and may depend upon the resources used.

There are resources other than print resources. Teachers are encouraged to employ various available computer programs, and calculators where appropriate. (TL)

Instructional Notes

Students can be introduced to the process of rationalizing by practicing selected exercises, and being asked to examine the results obtained in these situations.
E.g.:

$\sqrt{5} \bullet \sqrt{5} =$

$3\sqrt{7} \bullet 3\sqrt{7} \bullet 3\sqrt{7} =$

$(3\sqrt{2} - \sqrt{5})\ (3\sqrt{2} + \sqrt{5}) =$

They can be asked to determine an expression that could be multiplied by a given expression so that the result is a rational number.
E.g.:

$(4\sqrt{5} - 3\sqrt{7})\ (?) =$

a rational result.

Once the basic process of converting an irrational denominator to a rational denominator is observed by the students, they could be introduced to exercises which involve rationalizing the denominator. They could work on exercises individually, in pairs, in small groups, or as a class.

Students should be expected to summarize their results, and describe the processes utilized for denominators with square roots, and for denominators with cube roots. The teacher may wish to have students theorize about a process that could be utilized for denominators with other indices.

Examples/Activities

Some sample exercises are provided.

1. $\dfrac{5}{\sqrt{3} + \sqrt{2}}$

2. $\dfrac{3\sqrt{5}}{2\sqrt{3} - \sqrt{5}}$

3. $\dfrac{4\sqrt{3} + 5\sqrt{2}}{2\sqrt{3} + 2\sqrt{2}}$

4. $\dfrac{6\sqrt{2} - 4\sqrt{5}}{5\sqrt{3} + 2\sqrt{5}}$

5. $\dfrac{5}{\sqrt[3]{3}}$

6. $\dfrac{7\sqrt[3]{2}}{4\sqrt[3]{25}}$

Adaptations

Students could be asked to deal with denominators which include variables.
E.g.:

$$\frac{3\sqrt{x} - 4\sqrt{y}}{2\sqrt{x} + 5\sqrt{y}}$$

$$\frac{x\sqrt{3} - 2\sqrt{3}y}{2\sqrt{3}x + 5\sqrt{2}y}$$

This topic could also be extended by using binomial denominators with an index of 3.
E.g.:

$$\frac{3}{\sqrt[3]{7} - \sqrt[3]{4}}$$

$$\frac{x + y}{\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2}}$$

Objectives

D.8
To solve and verify the solutions of quadratic equations by factoring, completing the trinomial square, and using the quadratic formula.

Instructional Notes

Students have already solved quadratic equations by factoring, and by taking the square root of both sides of the equation, in Mathematics 20. As well, they have already reviewed these types previously in Mathematics 30 (see objective C.9).

Students should be given some practice in factoring perfect trinomial squares, and in determining the value of c in ax^2+bx+c, which would make ax^2+bx+c a perfect square. They could work cooperatively on these skills. Students should be able to summarize their findings, draw conclusions about the completion of a perfect square, and demonstrate their understanding by completing other examples. (CCT)

Once students are able to complete the square, this process can be utilized in solving quadratic equations. Students should observe some examples and work cooperatively to solve (and check) several examples of their own.

The quadratic formula can then be developed as an exercise in solving quadratic equations by completing the square for the general case $ax^2 +bx+c$. Students can attempt the general case in small groups, with the teacher providing assistance and hints as necessary.

Students should have time in which to practise using all of these methods in solving (and checking) quadratic equations.

Examples/Activities

1. Solve by factoring:

   a) $x^2+6x+8 = 0$

   b) $2x^2-5x+2 = 0$

   c) $3x^2 = 10x-3$

   d) $x+15 = 6x^2$

   e) $3x^2 -x = 2$

   f)* $3x^2-x = -2$

2. Solve by completing the square:

   a) $x^2-10x+9 = 0$

   b) $3x^2+12x-9 =0$

   c) $3x^2-5x+2 = 0$

   d) $16x^2-24x+5 =0$

3. Solve by using the quadratic formula:

   a) $x^2-6x-7 = 0$

   b) $6x^2+11x =10$

   c) $5x^2-8x = 20$

   d) $4x^2 +6x-9 = 0$

* You may wish to include some that do not have real number solutions.

Use graphic calculators or computers to graph each of the above. (change the 0 to y). Have students compare the graphs to the solutions. The visual representation may help student understanding. (TL)

Adaptations

This topic may be extended in many ways. Students could be asked to develop the quadratic formula on their own, given the general equation $ax^2+bx+c = 0$, and the method of completing the square.

Students could be asked to identify the relationships between the solution of the equation and the graph of the corresponding function.

All students should be presented with real-world situations where the solution of quadratic equations is necessary. These may be found in various resource texts, from the fields of engineering, commerce, health, agriculture, aeronautics, electronics, sports, and others. Successful solutions to these types of word problems will aid in the future study of maxima-minima problems in related classes. Students may be asked to search through school resources for some of these real-world applications, with the help of the resource teacher (librarian). (IL)

Most texts have several of these types of problems. The exercises in the left-hand column illustrate the types of equations that should be solved, but word problems that utilize each type of solution should be used.

Objectives

D.9
To solve and verify equations involving absolute value.

Instructional Notes

Review the meaning of absolute value as studied in Mathematics 20. There are several different methods of presenting this topic to students. This can be done as an exercise in solving equations, by designing a particular algorithm that may be used in all cases, by using the definition of absolute value and providing a visual representation, or by graphing when the 0 is changed to a y.

Students usually benefit if they are presented with several choices and can use one that most closely fits their learning style.

Most texts that include this topic use an algorithmic approach. This method is best suited to a lecture approach, where students work independently on the topic. Stress the fact that two solutions are to be found and both must be checked.

In the visual approach, the definition is used (e.g.: distance from zero). If the absolute value quantity is placed on a number line at the point that makes it zero, move the given distance to the left and to the right to obtain the two solutions (Example under Examples/Activities). Both must be checked in the original context. Students can work individually, in pairs, or in small groups.

A graphic approach can also be used, utilizing graphic calculators or computers. Students may work cooperatively. Solutions should be checked. (PSVS)

Examples/Activities

Solve each of the following absolute value equations.

1. $|x| = 5$

2.* $|x| = -3$

3.** $|x-4| = 3$

4.** $|3x-5| = 2$

5. $|5x+2| = 4$

6. $3|2x-1| + 8 = 20$

* No solution is possible, given the definition.

** Solution by visual method.

Solutions

3. Note absolute value quantity is (x-4). 4 is the value of x that makes this quantity equal to zero. Therefore, mark 4 on a number line. ______|______
   4

   Now the distance from this 'zero quantity' is equal to 3. On the number line, mark all distances of 3 from the 'zero quantity'.

   ______|_____|_____|______
   1 4 7

   Do 1 and 7 both check? These are the solutions.

4. Absolute value quantity is (3x-5). 5/3 is the value that makes this quantity equal to zero. This is marked on a number line. In this case, the distance is not just 2, but 2 divided by the coefficient of x, 3. The distance marked from the 'zero quantity' 5/3 is then 2/3.

   ______|_____|_____|______
   3/3 5/3 7/3

   Check if 1 and 7/3 are the solutions.

Adaptations

Students could search through resource materials to locate real-world situations that utilize absolute value. If any are found, they could be shared with the class. (Most applications are not direct, but rather absolute value tends to be an important part of the preliminary understanding of other concepts.) (IL)

Note that $|3x - 5| = 2$, therefore,
$3|x - 5/3| = 2$, and
$|x - 5/3| = 2/3$, hence 5/3 is the starting point, and 2/3 is the distance.

Objectives	Instructional Notes
D.10 To solve radical equations with two unlike radicands.	In Mathematics 20, students were introduced to radical equations involving one radical. A brief review of one or two exercises may be useful in introducing this topic. Most textbooks use an algorithmic approach to solving radical equations involving two radicals, and this approach may be utilized in the classroom. Students can be given a few instructions, and then be asked to complete a few exercises, working cooperatively in small groups or individually, to solve these exercises. After a short time, answers or possible solutions can be shared with the entire class. It should be pointed out that checking the solutions is important and the checking procedure should be carried out when solutions are presented to the class. Students should be asked to observe the solutions to determine if there are any general procedures that might be followed which make the solution to these equations simpler. Any such observations can be discussed with the entire class.
D.11 To solve word problems involving radical equations.	Once suitable word problems have been identified, students should work cooperatively to determine the necessary information, to pose the question to be answered and to write the equation, which, when solved, will provide the solution to the problem. (PSVS) Problems may be presented one or two at a time. Student answers may be shared with the entire class after students have had a few moments to work on the problems. A discussion of the main components of the problem, and the setup of the equation may help all students increase their skills in problem solving. A few problems may be assigned to the class to complete in small groups. Some statement should be elicited from the student as to whether the problem outlined in each case is a real-world situation, and if so, who might be responsible for its solution in its real-world context.

Examples/Activities

Solve each of the following equations. If an equation does not have a solution, please indicate.

1. $\sqrt{x+1} = \sqrt{x+4}$

2. $\sqrt{x-1} = \sqrt{x+4}$

3. $\sqrt{4x+3} - 5 = 2\sqrt{x-4}$

4. $\sqrt{4x+1} - \sqrt{x-2} = 3$

1. If one rectangle has dimensions of length $\sqrt{3x-2}$ and width $\sqrt{x}$, while a second rectangle has dimensions of length $\sqrt{x}$ and width 2, what must the dimensions be in order for the diagonals to have a difference of length 3 ?

Adaptations

This topic could be extended by having the students solve equations with indices other than 2. They could be presented with some equations involving cube roots, and others, and asked to solve and check their solutions.

They could also be asked to note any observations and conclusions about the solution of such equations. These could be recorded in a journal or daily log of their activities in mathematics. (COM)

Concept E: Relations and Functions

Foundational Objectives

- To demonstrate the ability to work with functional notation and related operations (10 05 01). Supported by learning objectives 1, 2, and 3.
- To be able to produce graphs of relations and functions and to be able to denote which graphs represent functions (10 05 02). Supported by learning objectives 4 and 8.
- To demonstrate the ability to interpret graphs representing functions and to identify key points of these graphs (10 05 03). Supported by learning objectives 5, 6, and 7.

Note: The Relations and Functions strand was first introduced in Mathematics 10, and continued in Mathematics 20. It is assumed that Mathematics A 30 students will continue their study of functions as outlined here. A review of some of the basic concepts of relations and functions can be found in Appendix B of this document, if the students need a refresher for some of the basic terms and concepts.

Objectives

E.1
To evaluate functions using functional notation.

Instructional Notes

This is the first occasion that the students will formally use functional notation. The teacher will find it worthwhile to introduce the f(x) notation, and have the students evaluate some examples where x is 2, 3, 4, for a simple function such as f(x) = 4x + 7. The students can then be instructed to graph y = 4x + 7, and look at their table of values. Draw comparisons between the values of f(x), where x is 2, 3, 4, etc., and the table of values. Repeat this with other functions familiar to the students.

Show how the ordered pairs for y = 3x - 5 can be generated by f(x) = 3x - 5, and the proper use of the notation.

Some students might have to be instructed that the technology they are using simply uses "y =" at all times, and this is the equivalent of f(x) for this purpose.

Alternatively, the teacher could use a discovery approach with the students, letting them draw the conclusions about y = and f(x) =. (CCT)

Examples/Activities

1. If f(x) = 2x + 8, then find the value of each of the following:
   a) f(3)
   b) f(-4)
   c) f(0)
   d) f(s - 3)

2. If f(x) = -2x + 1, then find the value of each of the following:
   a) f(-1)
   b) f(-4)
   c) f(6)
   d) f(2 + t)

3. If $g(x) = 3x^2 - 4x + 5$; find the value of each of the following:
   a) g(-2)
   b) g(3)
   c) g(-5)
   d) g(3 + h)

4. For h(x) = 4x - 3, and where x is -1, 1, 3, and 5, list the ordered pairs generated by h(x).

5. For f(t) = 2t - 8, what value(s) of t will make f(t) = 20?

Adaptations

Students should be introduced to some real-word situations where functional notation is involved. Examples such as distance as a function of time d(t), cost as a function of demand c(d), and the like could be used to introduce a variety of variables other than x. (NUM)

Objectives	Instructional Notes
E.2 To perform indicated operations on functions using functional notation.	Students might be asked to quickly define operations on numbers, and provide some examples. (Or the teacher could mention the operations, provide the examples, and have the students give the answers.) Then the students could be asked what they think might happen if they were to carry out these operations on functions and how they might go about it. As an example, they might be asked what would happen if they were to add the functions $f(x) = 2x + 3$ and $g(x) = -5x + 1$. In other words, what would result from $f(x) + g(x)$? Students could be asked to respond and to provide a rationale for their response. The teacher might have different groups of students attempt to use various methods, including technology. What would happen to this operation if we used the same value for x in each function? different values? Ask them to repeat their trials for another pair of functions and determine if their solution is justified. Have students explain various methods of solution to the class, and summarize. Other operations can be introduced and handled in a similar fashion. The teacher may summarize by showing the students accepted methods of calculation and their graphic representation.
E.3 To form the composite of two or more given functions.	Students can be asked to predict what might happen if one function is defined in terms of another function or functions. Display an example such as $f(g(x))$, where $f(x) = 6x - 1$ and $g(x) = 3x + 2$, and ask students to determine what might occur. They could discuss this in pairs, and attempt to resolve this by calculation, or with the aid of technology. (TL) Instruct them to use a specific value for x if they are experiencing difficulty. They then can be instructed to use a second value, and so on. Once the students seem to understand what is asked, several examples can be done in a classroom setting. Alternately, the use of mapping diagrams can be considered as a means of teaching this composition of functions.

Examples/Activities

Adaptations

1. For $f(x) = 2x - 3$ and $g(x) = 3x - 4$, find the value of each of the following:

   a) $f(2) + g(2)$
   b) $f(2) + g(3)$
   c) $f(-5) - g(3)$
   d) $f(4) - g(-1)$
   e) $g(3) - f(-5)$
   f) $f(3) \cdot g(1)$
   g) $f(3) \cdot g(3)$
   h) $4f(3)$
   i) $5g(2) - 3f(-2)$
   j) $g(3) \cdot 2f(3)$
   k) $f(m) + g(m)$
   l) $f(-2n) - g(n)$
   m) Find $(f + g)(x)$. Compare this notation to that used in a).
   n) Find $(f + g)(2)$.

1. If $f(x) = 2x^2 - x + 3$, and $g(x) = 3x + 1$, find the value of each of the following:
   a) $f[g(2)]$
   b) $f[g(-1)]$
   c) $2f[g(0)]$
   d) $4f[g(-2)]$
   e) $g[f(-1)]$
   f) $3g[f(0)]$
   g) $f[f(-2)]$
   h) $g[f(r)]$

Note that other styles of notation are used for functions in various resources. Students should be taught to realize that notations such as f o g, or f o g(x) are also acceptable to most mathematicians.

2. If $f(x) = 2x + 6$, $g(x) = -3x + 4$, and $h(x) = (1/2)x - 3$, find the value of each of the following:
   a) $f(3) + g(2) + h(1)$
   b) $g(4) - 2h(1) + f(5)$
   c) $g[h(2)]$
   d) $h\{g[f(3)]\}$
   e) $f[h(4)]$
   f) $h[f(4)]$
   g) What did you notice about e) and f)? Why did this happen? Would it happen for all functions? Try redoing part c) as $h[g(2)]$?
   h) Compare the results obtained for $f[g(x)]$ and $g[f(x)]$.
   i) $f(2 + h) - f(2)$
   j) $\frac{f(2 + h) - f(2)}{h}$

Concept E: Relations and Functions

Objectives

E.4
To determine if a function is one-to-one or many-to-one.

E.5
To write the equation of a line in standard form using: two intercepts, slope and one intercept, one point and the equation of a parallel line, and one point and the equation of a perpendicular line.

Instructional Notes

Students should be presented with the meanings of one-to-one and many-to-one functions, and then be asked to determine, with justification, which of a set of functions are one-to-one, and which are many-to-one. These functions should be presented as ordered pairs, mapping diagrams, and as graphs, in order for students to examine the relationships that exist between the three representations.

Alternatively, the teacher could present functions in one form and ask students to depict these functions in terms of the other two forms and then determine if they are one-to-one or many-to-one. (CCT)

Students should review the point-slope formula; e.g.:

$$\frac{y_1 - y_2}{x_1 - x_2} = m$$

In each case listed in Objective E.5, students should be able to graph using whatever method they wish. A copy of each graph should be kept on paper, in order to have a permanent copy. Analyzing each graph, the student should be able to determine the slope and a point (usually the y-intercept, but point out others) from each graph.

Small groups could discuss which are the important features that uniquely identify a graph (should be the slope and a point, or two points).

Have students go through each example, substituting the slope and one point(x_2,y_2) in the point-slope formula, and then simplifying the resulting equation. This will give students one method for writing any linear equation.

Examples/Activities

The students could be given a set of functions in any of the three forms they have studied, asked to place these functions in the other two forms, and determine if they are one-to-one or many-to-one.

The teacher may also ask students to generate types of functions which are one-to-one or many-to-one (or to locate examples from various resources).

Students may be expected to demonstrate some real life examples of the concept of one-to-one and many-to-one.
E.g.: (student in class, colour of eyes) should represent many-to-one.

I. Graph each of the following lines:

1. $m = 3$, $b = -2$
2. $m = 2/3$, $a = 5$
3. $a = -3$, $b = 5$
4. through (3,1) and parallel to $2x-y = 6$.
5. through (-2,5) and perpendicular to $3x+2y = 12$.

II. For each of the above, identify the slope and one point on the line.

III. Write the equation that represents the line described by each of the above.

Adaptations

Students can be given word problems where they must identify the slope and a point and then write the equation of the line so described. These problems can be taken from various resources or adapted from other sources.

E.g.: A student observes that a bouncing ball reaches a height of 180 cm on the first bounce, and on its subsequent bounces, loses 8 cm of height each time. What height will it be at on its 7th bounce, and what equation would best describe the bouncing pattern of this ball?

Objectives

E.6
To solve real-world problems involving quadratic functions by analyzing their graphs.

E.7
To graph quadratic functions of the standard form $f(x) = a(x-p)^2 + q$, by determining the vertex, axis of symmetry, concavity, maximum or minimum values, domain, range, and zeroes.

Instructional Notes

Students could work in small groups to solve problems based on quadratic functions. Students have previously worked with quadratic functions in Mathematics 20, so should have a basic familiarity with the shape of their graphs.

Each group should be instructed to graph the quadratic described in the problem (using whatever method they wish, e.g.: paper, calculator, computer), and then be instructed to determine the key components of the graph. They should decide which of these represents the answer to the problem they are to solve.

Their solutions can be shared with the class as a whole.

Have students using calculators, computers, or a table of values, graph several equations as written in the form in Objective E.7.

Using enlargements of the completed graphs, (on overhead or board), ask students to compare the key components of the graph with the equation as written. Have them note any comparisons between the key points and the equation and determine whether their observation is valid for all the examples.

When the students agree that a particular key component is represented by a part of the equation, ask them to note it in their journals or notes.

When all the major components have been agreed upon, the students can be given a few additional exercises to graph by hand, using the key components as identified from the equation. Each group can check their results against that of a calculator or computer.

Students should be expected to demonstrate their understanding of this technique by correctly identifying each of the key components listed in Objective E.7 and to sketch the graph of a quadratic equation written in this form.

Examples/Activities

Many of these types of problems can be found in resource texts. These can be adapted to utilize a graphic solution in most cases.

The teacher could introduce this section by having some prepared graphs drawn, and asking the students to determine the key components of the graphs. These key components might include the intercepts, the maximum or minimum points, the change in the ordinate from one point to the next, the domain and range.

In some introductory problems, the teacher might provide the graphs, and lead the class through a discussion on which of the components are important in a particular problem.

Graph each of the following:

1. $y = 2(x-3)^2+1$
2. $y = -3(x+4)^2-3$
3. $y = (1/2)(x-5)^2-4$
4. $y = (-2/3)(x+1)^2-2$
5. $y = 3x^2-5$
6. $y = (x-4)^2+5$
7. $y = -2(x+5)^2$

Adaptations

Have students identify occupations that might require employees to analyze graphs of quadratic functions occasionally. They may wish to first identify examples, and then relate these to the occupations. (IL)

The learning resource centre of the school may be used, or the guidance counselling/careers information centre for further search.

Students could be asked to graph an absolute value equation such as $y = 2|x-3|+1$ and compare the result to the one obtained for quadratics. Students might be instructed to graph several of these types of equations.

Students might also attempt to graph a quadratic non-function such as $x = 2(y-3)^2+1$, and observe the result.

Students/Teachers may prefer to use the form $y-p = a|x-q|$ as the vertex is then (p,q). The corresponding form for the quadratic might also be preferred.

Objectives / Instructional Notes

E.8
To graph quadratic functions of the general form $f(x) = ax^2+bx+c$, by completing the trinomial square and converting to one of the standard forms.

The process of completing the square was learned in the previous unit when solving quadratic equations.

The students could be presented with a quadratic equation such as $y = 2x^2-7x+6$, and ask them to work in groups to graph the equation in the form $y = a(x-p)^2+q$. Allow a few minutes for discussion. If little progress is being made, ask them to graph the equation on the calculator or computer, identify the key components, and generate the equation in standard form.

Students can be told to begin with the original equation and to utilize the process of completing the square to set the equation in standard form. They should be given some time to attempt this in small groups.

When some groups have been able to use the process to write the equation in standard form, they can demonstrate to the entire class how they were able to do this. The class may then be given a few other equations to write in standard form and graph.

E.9
To identify and graph examples of inverse variation taken from real-world situations.

Provide students with many examples of graphs, pictures, tables, statements, actual objects and mathematical symbols that can be used to illustrate inverse variation. Have students attempt to describe why each of these belongs to the group. (These examples could be pictures of gears or gears themselves, a graph, statements such as "the amount of gasoline burned by a vehicle depends upon its speed", etc.) (TL)

Students should define inverse variation first in their terms, and then in more formal terms.

Ask students to provide examples of inverse variations from their experience. Try to quantify some of these and have students graph the results.

Examples/Activities

Adaptations

Write each of the following equations in standard form and graph the result. Check your graph with the graph obtained by a calculator or computer.

1. $y = 6x^2-x-2$

2. $y = 4x^2+11x+6$

3. $y = 2x^2-7x+6$

For the following, graph only those which represent inverse variations.

1. An investor leaves her broker with the following instructions: "I wish to invest $500 per month in a mutual fund. For this, I would like a monthly statement giving me the number of shares purchased, and the cost of each share".

2. The foreman of a work crew must complete a specific job in a total of 288 worker-days. The superintendent wants to know the various possibilities for how many work days might be needed and how many workers are required.

3. The distance travelled by a vehicle depends on the time taken for the trip.

4. The time taken to complete a trip of 100 km depends upon the speed travelled.

5. The mass of an object depends upon the product of its density and its volume.

This topic might be modified by experimentation. A set of gears from the Industrial Arts lab, or from a service station might be used to illustrate inverse variation. (IL)

Alternatively, materials might be borrowed from the science lab to illustrate concepts such as illumination vs. distance, and weight-bearing vs. length.

Objectives

E.10
To state the domain and range, along with any restrictions, for the graphs of inverse variations.

Instructional Notes

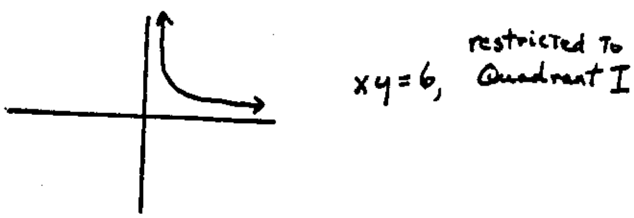
Present students with some graphs representing inverse variations. Have them state the domain and range of these examples. They should also be able to state restrictions to the domain and range.

Give students some inverse variations where they must both graph and identify the domain and range.
Many of these exercises could be taken from the previous objective.

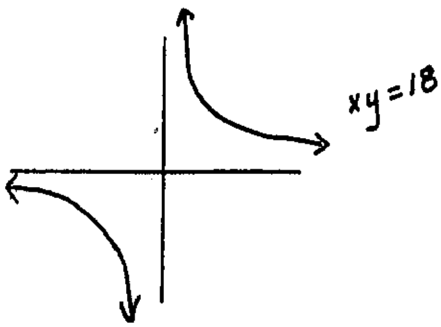
Examples/Activities

For each of the following graphs, state the domain and the range.

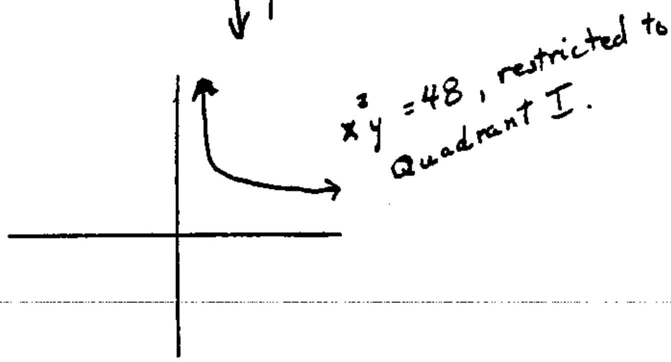
1.



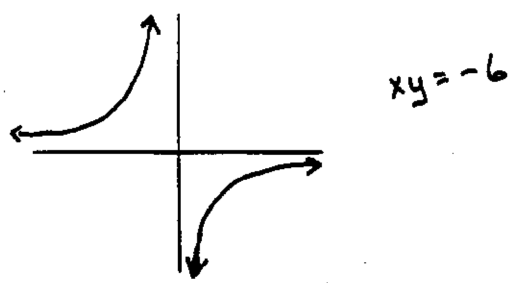
2.



3.



4.



Graph each of the following, and state the domain and range in each case. Discuss each answer with other group members.

1. $xy = 12$

2. The time it takes to travel 150 km at any speed from 10 km/hour to 160 km/hour.

Adaptations

Ask the students to provide a story example for each of the questions done in the previous column. Also have them determine who might be asked to provide this type of calculation or graph for their scenario.

Objectives	Instructional Notes
E.11 To determine the constant of proportionality of an inverse relation.	Provide the students with the definition of constant of proportionality and ask them to review several examples from work done in the previous objectives, to determine the constant in each case. They can share some of the examples they have used. The examples shared should cover mathematical statements and graphs. Provide students with a number of examples in which they are instructed to determine the constant of proportionality. Once students are able to determine the constant, have them work on related questions where the determination of the constant of proportionality is necessary to the solution of the question. Use real-world examples for problem solving situations of this type.
E.12 To solve problems that involve inverse variation.	Provide students with a sheet of problems and instruct them to determine which of these problems represents a situation that is an inverse variation. Once they have correctly sorted out the problems that represent inverse variations, instruct them to determine the constant of proportionality and to find the solution to the problem. In addition, you might have them graph these inverse variations and locate the solution on the graph. In the graph, what is the role of the constant of proportionality?

Examples/Activities

Identify the constant of proportionality in each case.

1. $xy = 8$
2. $D = m/V$
3. The pairs of factors of 48 are.....
4. The length of a beam is inversely proportional to its strength.
5. $x \propto 1/y$

Calculate the constant of proportionality in each of the following.

1. $x \propto 1/y$, where x is 6 when y is 8.
2. $x(y-2) = k$, when x is 5 and y is 10.
3. $x \propto 1/y^2$, where x is 4 when y is 8.

Calculate the indicated value in each case.

1. If $x \propto 1/y$, and x is 12 when y is 6, what is the value of x when y is 9?
2. If $x \propto 1/y^2$, and x is 3 when y is 6, what is the value of y when x is 2?
3. If the strength of a beam is inversely proportional to its length, and a beam 5 m long can support at most 100 kg, how many kg can a beam of 8 m support?

There are examples of real-world inverse variation problems in most textbooks. These can be used by the classroom teacher. (IL)

Students could work in pairs or small groups to discuss and solve these problems. (PSVS)

Adaptations

Have students discuss other methods they might use in employing the constant of proportionality. (E.g.: as a proportion statement they might have used in science.) Instruct them to use one or two of the examples already done and to complete them by using proportions. (COM)

Concept F: Systems of Linear Equations

Foundational Objectives

- To be able to identify the number of possible solutions of a system of linear equations (10 06 01). Supported by learning objective 3.
- To demonstrate the ability to solve a system of linear equations (10 06 02). Supported by learning objectives 1, 2, and 4.

Objectives

F.1
To solve and verify systems of linear equations in two unknowns by the following methods: graphic, substitution, and elimination.

Instructional Notes

This section may take several class periods to cover completely. It may be introduced by dealing with one method at a time, the typical textbook approach, or it may be somewhat more integrated.

As students have already done some graphing, it may be useful to begin with this method. Students should discuss the various possibilities that exist when two lines are drawn in the same plane. What are the possible relationships between the two lines?

Graphing may be done on a calculator or computer, so that students can quickly determine that the point of intersection is the most useful, if one exists. By using this technology, they can quickly solve many different examples. Examples should be varied to let students experience several cases where parallel lines, and the same line, also occur.

By working in small groups, students should be able to distribute the workload so that each member can use a different technique to solve the same problem. For each, they can compare to determine which is the most efficient method for that particular question. Students should rotate techniques, to practice several of those listed. (PSVS)

You may wish to have students graph each question on calculators/computers after they have used another method, as a visual check.

F.2
To solve linear systems in two unknowns that have rational coefficients and to verify the solutions.

Provide students with some examples of such systems of equations. Have them work together to discuss how they might find the solution to these systems. Ask the students for their suggestions on the approach to be used, making a list of various strategies suggested by the groups.

Assign each group one or two strategies to use in solving these systems. Have successful approaches presented to the class. (Some students may graph, others may use a variety of mathematical methods.)

If students are unsuccessful, the teacher may show some different techniques for sample purposes.

Students should be able to complete several of these types of systems.

Examples/Activities

An introductory activity might be to ask the class to work in small groups to attempt to solve the following problem in any manner: (Allow them to use calculators, computers, pencil and paper, graphs in their groups.)

The general manager of the Zoren Circus is to determine how many tickets need to be sold to ensure a matinee performance will make a profit for the circus. The manager knows from past experience that the number of children under 12 attending a matinee will be four times the number of adults attending. The total expenses associated with a matinee are $689. If adult tickets are $5 and children's tickets are $2, how many of each must be sold to break even?

Solutions arrived at should be shared with the class, justified, and discussed.

The questions assigned could be done in a standard format:

1. Solve by graphing
2. Solve by substitution
3. Solve by elimination

Each of the above would have to be introduced to the students, either through their solutions to the practice problem or by teacher example.

Solutions 2 and 3 could be checked by graphing.

Adaptations

To adapt this concept for less able students, the teacher might simply ask them to solve a word problem that involves two unknowns using any method they wish. Once their solutions are shown to be correct, equations can be made to outline the situation and the concept can be formalized.
E.g.:
Celine has nothing in her purse but seven coins, all quarters or dimes, that add up to a total of $1.15. How many of each does she have?

Most students are able to solve this by using the 'guess and check', or the substitution method. When they determine the answer, they could be instructed to write equations representing the situation, and asked to solve in a more formal manner. (NUM)

Solve each of the following systems, using any of the methods you have learned. Check your answer by using a different method.

1. $1/4\ x + 1/3\ y = 4$
   $1/2\ x - 5/6\ y = -1$

2. $3/2\ x + 2/5\ y = 13$
   $1/5\ x - 2/3\ y = 16/3$

Students should summarize the techniques employed and note these in a daily journal.

Concept F: Systems of Linear Equations

Objectives	Instructional Notes
F.3 To recognize the characteristics of linear equations in two variables with graphs that are inconsistent, consistent-dependent, or consistent-independent.	Provide students with several examples of each type of system. Instruct them to graph all these systems, and determine the solution. (If there is no solution, ask them to describe the situation.) When the systems have all been graphed, ask students to classify each system based on the number of solutions. When the graphs have been classed as having one, none, or many solutions, provide the students with the appropriate definitions for each, as listed in Objective F.3 Provide the students with several more systems, asking them to work in their groups to classify each system without graphing it. Have them discuss their justifications with other group members. (PSVS) If the entire class is satisfied with the shared results, they can be summarized as the characteristics of these systems. If the groups are not able to satisfactorily conclude how these graphs can be classified, the teacher should provide clues.
F.4 To solve word problems involving linear systems in two variables.	Provide students with two or three real-world problems that involve systems of linear equations. Try to supply problems that are somewhat different in nature. In groups, have students attempt to solve these problems by writing the system of equations needed, and then finding the solution, if one exists. (Students may be instructed to graph their equations.) Most textbooks have adequate exercises on this topic, but additional real-world examples may be gleaned from polls reported in newspapers, and from various magazines and journals. (IL)

Examples/Activities

By graphing, determine whether each of the following systems has zero, one, or many, solutions.

1. $3x-4y = 8$
   $9x = 12y + 4$

2. $2x-5y = 12$
   $1/2\ x = 5/4\ y + 3$

3. $4x+1/3\ y = -1$
   $2x - y = -5$

Classify, by sight, each of the following systems as inconsistent, consistent-dependent, or consistent-independent.

1. $4x+5y = 20$
   $10y - 40 = -8x$

2. $2x-3y = 12$
   $-4x-5y = -2$

3. $3x-2y = 7$
   $-6x + 3y = -21$

Solve each of the following problems in your groups, with group members using different methods, as a means of checking the solution.

1. Joe remembers that a certain NHL hockey player set a record for most points in a season with 215. (Points are found by combining the number of goals and the number of assists.) He also remembers that the player obtained 111 more assists than goals. How many goals did the player have that year?

2. Washers at $650 each, and dryers at $500 each were sold during a year at a local appliance store. Invoices showed that the total number of machines sold was 445, and the total receipts were $236 250. How many of each were sold?

3. A fisheries biologist obtains a test netting that catches 14 more than three times as many pickerel as pike. Altogether, there were 130 fish caught in this test netting. How many of each were caught?

4. A mixed-farming operation has both cattle and chickens as well as grain. One day, the owner notices that the chickens and cows have a total of 162 eyes and 258 legs. (Obviously, the owner doesn't have much to do.) How many of each are there?

Adaptations

This topic could be adapted by starting with a set of graphs of pairs of lines. The students could be asked to separate these graphs according to some of the characteristics of the graphs. (Hopefully, these categories would be those of consistent, consistent-dependent, or inconsistent.) The equations of each system could be listed under its category and analyzed in this fashion.

Students could research various resource texts for different problems of this type and classify them as political, economic, sports, etc. This might help some students relate mathematical concepts to the real-world.

Concept G: Angles and Polygons

Foundational Objective

- To demonstrate the ability to determine trigonometric ratios in a given situation, and apply these ratios to solving real-world problems (10 07 01). Supported by the following learning objectives.

Objectives	Instructional Notes
G.1 To sketch an angle in standard position.	Students should have graph paper, straight edge, protractor, and pencil. Instruct them to draw several sets of axes on a sheet of graph paper (4 or 6). When this task is complete, instruct them to draw an acute angle on their first graph. Share these with the class by quickly observing other group members' graphs. Note how many have one side of the angle parallel to the x-axis, how many have one side on the x-axis, and how many have one side on the x-axis and the vertex at the origin. Ask how many have none of the three above conditions. Introduce the concept of standard position of an angle, and have students draw specific angles (e.g.: 30°, 45°, 77°, 125°) in standard position on the other sets of axes they have prepared.
G.2 To determine the distance from the origin to a point on the terminal arm of an angle in standard position.	Students should have graph paper, a ruler, and a pencil. Instruct them to draw several sets of axes on a sheet of graph paper. Label the origin O in each case. Give them a specific ordered pair to plot (3,4), and label it A. Ask them to determine the distance from O to A. They may measure or use the Pythagorean Theorem. Have them do several such examples. Share the solutions with the class having students describe the method used in each case. Discuss which method seems to be the most efficient. The teacher may summarize by using standard trigonometric nomenclature for the legs and hypotenuse of the Pythagorean Theorem, reducing this to $x^2+y^2 = r^2$. Have students do one or two exercises using this formula.

Examples/Activities

Sketch each of the following angles in standard position:

1. 25°
2. 15°
3. 225°
4. 325°
5. 180°

1. Determine the distance from the origin to the given ordered pair in each instance.
   a) (5,12)
   b) (-4,3)
   c) (-3,-2)
   d) (5,-2)
   e) (0,-6)
   f) (7,0)

Adaptations

Students could be provided with a series of angles drawn in various locations on a set of axes, and instructed: a) to identify those that are in standard position and measure them, and b) to transpose those that are not in standard position to standard position.

Concept G: Angles and Polygons

Objectives	Instructional Notes
G.3 To calculate the distance between two ordered pairs in the coordinate plane.	A few exercises involving the Pythagorean Theorem could be used as a starting point. The Pythagorean Theorem was introduced in Mathematics 10. These exercises would also serve as a visual reminder to students. Students could be instructed to draw triangles on graph paper, such that one end of the hypotenuse was at the origin. (The ordered pairs representing the vertices should be integers.) Students can then be instructed to calculate the length of the hypotenuse. Have students work in small groups to discuss the procedure they would use to determine the length of the hypotenuse of a right triangle that is not drawn at the origin. Have them draw such triangles on graph paper (ordered pairs representing vertices should be integral values with the two legs of the triangle parallel to the axes) and determine the length of the hypotenuse. When students have been able to calculate the hypotenuse, with justification, have them determine the length of the hypotenuse for the general case (legs parallel to the axes). Once the class has developed a procedure for determining the hypotenuse for the general case, ask students to work in their groups to determine the distance between two given points (some groups may need additional help in starting). (PSVS) When students are able to find this distance, they can be given the distance formula (standard form) to use.
G.4 To determine the coordinates of the midpoint of a segment.	Ask students to discuss the meaning of midpoint. Introduce a number line, and ask the class to identify the midpoint of two specific points. Have students complete several examples. Repeat using a vertical number line. Introduce two ordered pairs on a graph, and ask students to determine the midpoint of these two ordered pairs. After several examples, ask students to determine the midpoint of the ordered pairs (x_1,y_1) and (x_2,y_2). The completion of this exercise should result in the formula for obtaining the midpoint of any two ordered pairs.

Examples/Activities

A set of exercises could include examples similar to the following:

1. Draw a few triangles with measurements given for the two legs, and instructions to calculate the length of the hypotenuse.

2. Draw a few right triangles on a graph such that one end of the hypotenuse is at the origin and give the ordered pairs representing the vertices. Instruct students to calculate the length of the hypotenuse.

3. Draw some right triangles on a graph, such that the legs are parallel to the axes and give the ordered pairs representing the vertices. Instruct students to calculate the length of the hypotenuse.

4. Give students two ordered pairs to plot. Instruct them to use these to construct a right triangle (with the given pairs representing the endpoints of the hypotenuse) and calculate the length of the hypotenuse.

5. Instruct the students to calculate the distance between pairs of given points. Calculate the distance from A to B in each of the following:
   a) A (3,5); B(6,9)
   b) A (-2,4): B(3, -8)
   c) A(4,-3): B(-1,4)

6. Give some word problems that incorporate the distance formula.
   a) A surveyor wishes to determine the actual distance from a rocky outcrop to a dock on the opposite side of a northern lake, for purposes of stringing a power line. On a map these points are labelled as (4.5 km, 3.1 km) and (5.0 km, 1.9 km). What is the distance between them?

1. Determine the midpoint of the line segment joining the following pairs of points:
   a) (5,1), (13, 7)
   b) (-4, 7), (6,-3)
   c) (5,-4), (-6, 9)

2. On a map with numerical references, the village of Sundown is located at (6.3 km, 2.9 km), while the town of Sunup is located at (4.7 km, 13.2 km). A water line is to be constructed between the two centres. Each community will be responsible for the construction of the line to the midpoint. At which point do they meet? If Sundown's projected costs are estimated to be $63,475 per kilometre, what would be their share of the construction cost? (NUM)

Adaptations

Students could be expected to find other real-world examples of uses of the distance formula by reviewing other resources. The teacher may also wish to point out to the students that many of the identities used in Trigonometry are based on the distance formula (Pythagoras' Theorem). (IL)

Alternatively, word problems of the comparison type might be used to check students' understanding of this concept.
E.g.: Which is the greater distance; from (6,5) to (-1,3), or from (-5,-3) to (0,-8)? By how much ?

The use of software with graphing capability will add technological skills to mathematical competence.

Concept G: Angles and Polygons

Objectives	Instructional Notes
G.5 To determine the value of the six trigonometric ratios when given a point on the terminal arm of an angle in standard position. (x,y,r)	In Mathematics 10, students learned to find the three primary trigonometric ratios using a right triangle, and the concepts of hypotenuse, opposite sides, and adjacent sides. This can be quickly reviewed and utilized as a starting point for this objective. Have students draw several sets of axes on a sheet of graph paper and plot a specific ordered pair on the first axes. Instruct them to determine the distance of the ordered pair from the origin. Using the idea of rotation in a circle and the angle in standard position, x becomes the adjacent side, y the opposite side, and r the hypotenuse. Redefine the three basic trig functions in terms of x, y, and r. Have students determine the values of these three basic functions using x, y, and r. Introduce the functions of secant (r/y), cosecant (r/x), and cotangent (x/y) as reciprocals of the basic functions. Have students find the values of these functions as well. Assign several other ordered pairs where students are asked to determine all six trigonometric functions of an ordered pair. Use ordered pairs from different quadrants and some whose coordinates include a zero. Ensure that each student sees that for angles where $\theta < 90°$, $\sin \theta = \frac{opp}{hyp}$ and $\sin \theta = \frac{y}{r}$, are the same.
G.6 To determine coterminal angles for a given angle.	Have students draw an angle in standard position, and denote it by marking it with a curved line drawn from the initial side to the terminal side. Measure this angle with a protractor and record. Have the students discuss the existence of any other angles that might have the same initial and terminal sides. (The concept of a negative angle should be introduced in this section.) Students should be able to find both the negative coterminal angle and the positive angles that are coterminal. Conclusions can be shared with the entire class and summarized for the entire class. Instruct students to determine at least 6 coterminal angles for a specific case (E.g.: 50°), three negative and three positive. Repeat with one or two other angles. Then have them discuss in groups how they might represent this in a general form. (Same e.g.: $50° + n \cdot 360°$, where n is any integer). Have students do a few examples of the general case.

Examples/Activities

Adaptations

Determine the six trigonometric ratios for each of the following ordered pairs, where the point represents the end of an arm of an angle in standard position.

1. (5,12)
2. (-3,4)
3. (-8,-15)
4. (2,-3)
5. (-5,0)
6. (0, 7)

---

1. Determine three positive and three negative coterminal angles for each of the following.
   a) 60°
   b) 125°
   c) 305°
   d) 180°
   e) -75°
2. Determine the general form of the coterminal angles for each of the following.
   a) 45°
   b) 173°
   c) -215°

Concept G: Angles and Polygons

Objectives	Instructional Notes
G.7 To determine the reference angle for positive or negative angles.	Since most tables used for trigonometry tended to give values between 0 and 90 degrees, it used to be the case that students would have to use the reference angles to find the trigonometric values for any angle greater than 90 degrees. Most calculators are now able to give the trigonometric values for any angle. The reference angle is no longer necessary for this type of operation; however, in the cases of exact values, it may still be necessary to have students deal with reference angles. Have students plot the ordered pairs (3,4), (-3,4), (-3,-4), and (3,-4) on a graph, and determine the six trigonometric values of each. Have them measure the angle between the x-axis and the ordered pair (with vertex at the origin) with a protractor. Give them another set of four ordered pairs and ask them to repeat the exercise. Discuss the outcomes in small groups and determine how one might calculate such reference angles (reference $\angle = \mid \theta$ - nearest ray of x-axis$\mid$). Have them find several reference angles.
G.8 To determine the values for the six trigonometric ratios, when given one trigonometric ratio and the quadrant in which the angle terminates.	Students can be given a trigonometric ratio, and the quadrant, and asked to determine the other five ratios. They could be instructed to plot the given information on a graph, to find the necessary missing information, and to provide the remaining trigonometric ratios. If student groups experience difficulty, the teacher might suggest that they try to identify x, y, and r, and use the process outlined under Objective G.5. Solutions can be shared with the other groups in the class. Provide a few exercises of the same type for students to obtain practice.

Examples/Activities

Find the reference angle of each of the following.

1. 140°
2. 215°
3. 345°
4. 97°
5. 272°
6. 175°

Determine the six trigonometric values of each of the following, using reference angles in each case.

1. 150°
2. 255°
3. 330°

Determine the values of all six trigonometric ratios, given the following information.

1. $\sin \theta = 4/5$, θ in Q.II
2. $\cos \theta = -2/3$, θ in Q.IV
3. $\tan \theta = 5/6$, θ in Q.III
4. $\csc \theta = -13/5$, θ in Q.III

Adaptations

Students could be asked to provide the six trigonometric values for the angle formed by a line segment through the origin, and in a specific quadrant. E.g.: Determine the six trigonometric values for the angle in standard position formed by $3x-4y = 0$, in the third quadrant.

This is a good time to redefine the slope of a line as the tangent of its angle of inclination (in standard form).

$m = \tan \theta \quad 0° < \theta < 180°$. Why?

Concept G: Angles and Polygons

Objectives

Instructional Notes

G.9
To determine the values for the trigonometric ratios by using a calculator.

Given a trigonometric ratio, students should use a calculator to determine the angle associated with that ratio.

Provide several exercises of the type sin θ = .6336, cos θ = .9200, tan θ = -1.385, and instruct students to determine all possible values (0°< O< 360°) with which these trigonometric ratios are associated.

Present students with a set of more specific exercises such as sin θ = .7335 in Q.I, cos θ = -.3882 in Q.II, tan θ = 3.247 in Q.III and instruct students to determine the angle using calculators.

As a third type of exercise in this section, have students draw and label two or three right triangles, and assign lengths for any two sides. Then instruct the students to determine the size of the acute angles in each triangle by setting up the proper ratios, and using calculators to determine the angle.

G.10
To apply the trigonometric ratios to problems involving right triangles.

Present students with a real-world problem to discuss and solve in their group. They should have pencil, paper, straightedge, and calculators in order to illustrate the problem and solve it.

The problems presented to the students should be of two types, the first where the sides are given and they are instructed to determine the angle(s), and the second where a trigonometric ratio of an angle is given, and they are instructed to find the side(s) or the angle(s).

Most resource texts have an adequate number of word problems of this type. (IL)

Examples/Activities

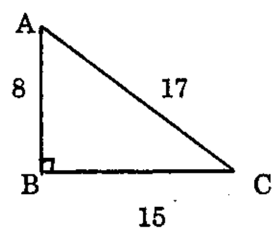
Determine all possible values between 0° and 360° for θ in each of the following.

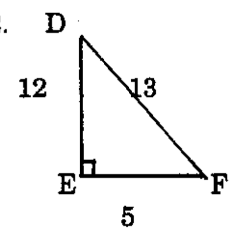
1. $\cos \theta = .7660$
2. $\csc \theta = -1.9876$
3. $\cot \theta = -2.3548$
4. $\sec \theta = 1.8182$
5. $\sin \theta = -.3486$
6.* $\sin \theta = 2.3500$

Determine the value of θ in each of the following.

1. $\sin \theta = .4529$, in Q.IV
2. $\cos \theta = -.6525$ in Q.III
3. $\tan \theta = 1.100$ in Q.I

Determine the size of the acute angles in each of the right triangles below.

1. 

2. 

1. Jodi has a kite which is attached to a 100 m cord. When flying the kite one day, Jodi notices that the kite has reached the end of this cord. Brayden, standing 72 m away from Jodi, states that the kite is directly overhead. How high is the kite and what is the angle the cord makes with the ground?

2. A lawnmower has a handle 1.5 m long and is attached to the lawnmower at a point 20 cm above the ground. The manual states that the maximum efficiency in the handling of the lawnmower is reached when the handle is at an angle of 60^0 with respect to the ground. How high off the ground must the handle be held to achieve this maximum efficiency?

3. A carpenter wishes to install a shelf at one end of a hallway. Because of space restriction, the shelf must be a right triangle with legs of 75 cm and 100 cm. The carpenter determines the angle formed by the 75 cm leg and the hypotenuse in order to set the saw correctly. What is the size of the angle?

Adaptations

Students can be presented with problems where the solution to the problem depends upon solving more than one triangle. They can work in their groups, making sure to draw sketches of the situation as described in the problem, and discussing various approaches to solving the problem. E.g.: A receiving antenna 5 m tall is installed on the edge of a roof of a building. The installer notes that from a point on the street 50 m from the building, the angle to the top of the building is 42°and that the angle to the top of the antenna is 46°. How tall is the building? (TL)

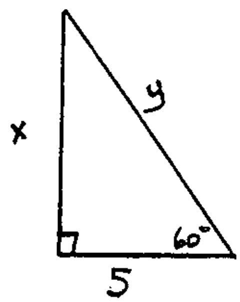
Objectives	Instructional Notes
G.11 To determine the relationships among the sides of each special right triangle (45°-45°-90° and 30°-60°-90°).	Have the students construct isosceles right triangles where the legs have integral measures, and measure the hypotenuse. Then have them calculate the length of the hypotenuse using the Pythagorean Theorem. They can compare their results with the results of other group members. Have them draw other isosceles right triangles and repeat the process. In their groups, have them discuss their results and summarize them for sharing with the entire class. Ask the students to generalize their results for an isosceles right triangle whose legs are of length x. In a similar fashion, ask them to determine the relationships of the legs and the hypotenuse of a 30°-60°-90° triangle. Students should summarize the results from both types of triangles and understand the relationships for the general case.
G.12 To calculate the length of the missing sides of the special right triangles when given the exact value of one side.	Once students become familiar with these relationships, they can be given exercises which require them to calculate the lengths of the missing sides based on these relationships.

Examples/Activities

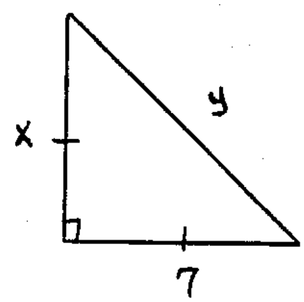
Adaptations

Calculate the lengths of the missing sides in each of the following cases.

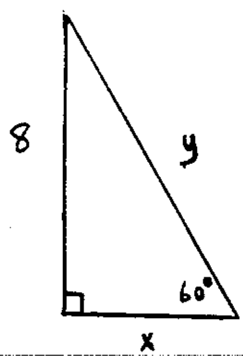
1.



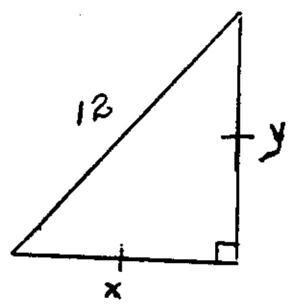
2.



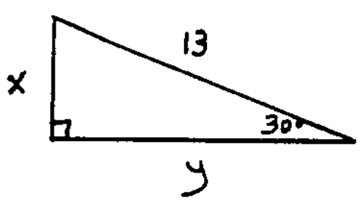
3.



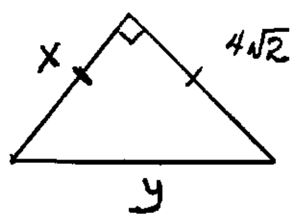
4.



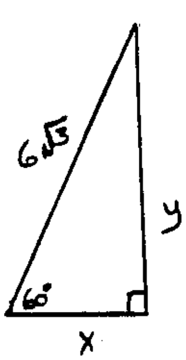
5.



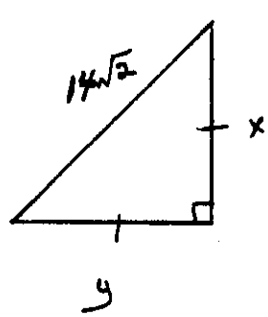
6.



7.



8.



Mathematics B 30

Western Protocol - Common Curriculum Framework (1996)

10-12 Mathematics - General Outcomes

Number (Number Concepts)

- Analyze graphs or charts of given situations to derive specific information.
- Analyze the data in a table for trends, patterns and interrelationships.
- Explain and illustrate the structure and the interrelationship of the sets of numbers within the real number system.
- Explain and illustrate the structure of the complex number system and its subsets.

Number (Number Operations)

- Use basic arithmetic operations on real numbers to solve problems.
- Describe and apply arithmetic operations on tables to solve problems, using technology as required.
- Describe and apply arithmetic operations on matrices to solve problems, using technology as required.
- Make and justify financial decisions.

Patterns and Relations (Patterns)

- Represent naturally occurring discrete data, using linear or nonlinear functions.
- Generate and analyze number patterns.
- Investigate the nature of mathematical reasoning.
- Generate and analyze recursive and fractal patterns.

Patterns and Relations (Variables and Equations)

- Generalize operations on polynomials to include rational expressions.
- Represent and analyze situations that involve variables, expressions, equations and inequalities.
- Use linear programming to solve optimization models.
- Solve exponential, logarithmic and trigonometric equations.

Patterns and Relations (Relations and Functions)

- Examine the nature of relations with an emphasis on functions.
- Represent by models naturally-occurring data using linear functions.
- Represent and analyze functions using technology, as appropriate.
- Use the concept of function to solve problems.

Shape and Space (Measurement)

- Use measuring devices to make estimates and to perform calculations in solving problems.
- Demonstrate an understanding of scale factors, and their interrelationship with the dimensions of similar shapes and objects.
- Solve problems involving triangles, including those found in 3-D applications.
- Analyze objects, shapes, and processes to solve cost and design problems.

Shape and Space (3-D Objects and 2-D Shapes)

- Solve co-ordinate geometry problems involving lines and line segments.
- Develop and apply the geometric properties of circles and polygons to solve problems.
- Classify conic sections, using their shapes and equations.
- Solve problems involving triangles and vectors, including 3-D applications.

Shape and Space (Transformations)

- Perform, analyze and create transformations of functions and relations.

Statistics and Probability (Data Analysis)

- Describe, implement and analyze sampling procedures and draw appropriate inferences from the data collected, using mathematical and technical language.
- Apply line-fitting techniques to analyze experimental results.
- Analyze bivariate data.

Statistics and Probability (Chance and Uncertainty)

- Make and analyze decisions using expected gains and losses based on single events.
- Model the probability of a compound event in order to solve problems based on the combining of simpler probabilities.
- Solve problems based on the counting of sets, using techniques such as the fundamental counting principle, permutations and combinations.
- Use normal probability distribution to solve problems involving uncertainty.

Mathematics B 30

A. Probability

Foundational Objectives

- To demonstrate the ability to set up and calculate probabilities of related events (10 01 01). Supported by learning objectives 1 to 5.
- To apply the Binomial Theorem to expand binomials, and to real-world problems (10 01 02). Supported by learning objectives 6, 7, and 8.

B. Data Analysis

Foundational Objectives

- To determine the standard deviation of a set of data, and to utilize the standard deviation in analyzing that set of data (10 02 01). Supported by learning objectives 1 to 3.
- To develop skill in interpreting data through the use of z-scores (10 02 02). Supported by learning objectives 4 to 6.

C. Matrices

Foundational Objectives

- To illustrate appropriate real-world situations using matrices (10 03 01). Supported by learning objective 2.
- To demonstrate knowledge of terms associated with matrices (10 03 02). Supported by learning objectives 1 and 6.
- To develop skills in matrix operations and in solving related real-world problems (10 03 03). Supported by learning objectives 3, 4, 5, 7, 8, 9, 10, 11, and 12.

D. Complex Numbers

Foundational Objective

- To demonstrate the skills developed in operations with complex numbers (10 04 01). Supported by the following learning objectives.

E. Quadratic Equations

Foundational Objectives

- To demonstrate skill in solving quadratic equations (10 05 01). Supported by learning objectives 1, 2, 3, 7, and 8.
- To write a quadratic equation through analysis of the given roots (10 05 02). Supported by learning objectives 4, 5, and 6.

F. Polynomial and Rational Functions

Foundational Objectives

- To demonstrate the ability to graph and to analyze the graphs of polynomial and rational functions (10 06 01). Supported by learning objectives 1 to 3.
- To demonstrate understanding of an inverse of a function (10 06 02). Supported by learning objectives 4 and 5.

G. Exponential and Logarithmic Functions

Foundational Objectives

- To develop skills and knowledge in working with a variety of exponential and logarithmic functions (10 07 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to apply the knowledge of exponential and logarithmic functions to real-world situations (10 07 02). Supported by learning objectives 6 to 17.

Concept A: Probability

Foundational Objectives

- To demonstrate the ability to set up and calculate probabilities of related events (10 01 01). Supported by learning objectives 1 to 5.
- To apply the Binomial Theorem to expand binomials, and to real-world problems (10 01 02). Supported by learning objectives 6, 7, and 8.

Objectives

A.1
To define the principle of inclusion and exclusion when working with two or more sets and/or events.

Instructional Notes

Some students may have been introduced to this concept previously (See Mathematics A 30 - concept A. 1). This concept is needed before beginning the section on probability.

Students can be introduced to the concept of inclusion and exclusion by dealing with two sets of objects at the outset, and then expanding this to three or more sets of objects. Students should work individually, or in pairs, with some initial exercises designed to illustrate the principle of inclusion and exclusion.

Venn diagrams should be used to illustrate these concepts. When students have become familiar with the concepts, formal notation should be introduced.
E.g.: $|A \cup B| = |A| + |B| - |A \cap B|$, or
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C|$
$- |B \cap C| + |A \cap B \cap C|$.

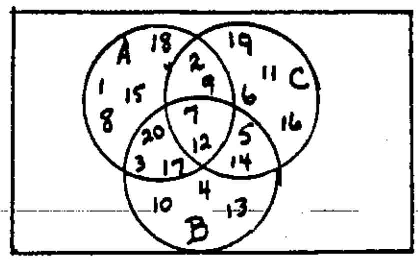
Examples/Activities

1. a) Given the sets A = { 2,3,4,5,6,7} and B = {1,3,5,7,9}, list the elements these sets have in common.
   E.g.: what is A ∩ B?

   b) For these sets A and B, list all the elements that are present. E.g.: what is A ∪ B?

   c) Illustrate both A ∪ B and A ∩ B by drawing a Venn diagram.

2. Given the following Venn diagram, identify the numbers represented by each of the following;
   a) |A ∪ B| b) |A ∪ C| c) |B ∪ C|
   d) |A ∩ B| e) |A ∩ C| f) |B ∩ C|
   g) |A ∩ B ∩ C|



3. Draw a Venn diagram to illustrate the following situation, and then solve the problems.

   Ms. Jones is the Vice-Principal of a comprehensive high school in Saskatchewan. Part of her job is to determine the timetable for the school year. From the Grade 12 student request forms this year, she has found that 163 students requested Mathematics, 95 requested Physics, and 124 requested Chemistry, while 17 students requested none of these. As well, she noted that 61 requested Mathematics and Physics, 67 Mathematics and Chemistry, 49 Chemistry and Physics, and 27 requested all three. How many requested only Mathematics? Only Physics? Only Chemistry? How many request forms (students) were there?

Adaptations

Students having difficulty with these concepts could be given sets of playing cards and asked to model situations of the type that illustrate these concepts. As an example, find the sets of cards that would illustrate:
a) aces or kings
b) aces or spades
c) face cards or diamonds

For each of these situations, the principles of inclusion and exclusion can be discussed and written down.

Other sets of manipulatives could be used instead of playing cards.

Students might also be instructed to attempt some questions of a more abstract variety, such as the following:

If A ∩ B contains 17 elements, while A itself has 46 elements, and B has 39 elements, how many elements are in A ∪ B? (CCT)

Concept A: Probability

Objectives

A.2
To determine the probability of mutually exclusive events.

Instructional Notes

The term mutually exclusive should be defined and examples provided by/for the students to indicate their understanding of the term. When this is complete, students may attempt some exercises that involve mutually exclusive events. They should work individually, in pairs or in small groups to determine the solutions to these questions. A brief review of the concept of probability may also be necessary. (PSVS)

Some examples they might work on are as follows:

a) Given two Canadian pennies, find the probability of tossing two heads or two tails.
b) Given a set of cards numbered from 1 to 6, find the probability of drawing a 5 or a 6.

Exercises should become progressively more difficult, as the students' understanding deepens. Students should be able to determine the pattern for calculating the number of outcomes for mutually exclusive events, determine if A and B represent mutually exclusive events, then note that P(A or B) = P(A) + P(B).

It is important to note that for any two events that are not mutually exclusive, we have $P(A \cap B) \neq 0$. The teacher may wish to include some questions of this type as well.
For these, the $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, which can be related to the principle of inclusion and exclusion.

E.g.: What is the probability of rolling a 4 or rolling an even number?
E.g.: What is the probability of drawing a king or a red card?

Examples/Activities

1. What is the probability that a card drawn at random from a standard deck of 52 playing cards will be a king or a queen?

2. If a die is rolled, what is the probability it is a 5 or a number less than 3?

3. When rolling two dice at the same time, what is the probability of obtaining a total of 7 or a total of 11 on one roll?

4. Statistics indicate that 64% of our population lives in cities, 27% in suburban areas, and 9% in rural areas. If we are to randomly select a person to take part in a poll, what is the probability that person will be from a rural or suburban area?

5. If you toss a coin and roll a fair die, what is the probability of tossing a head or rolling a 3?

6. What is the probability of drawing a face card (jack, queen, king) or a club?

Adaptations

The teacher should introduce this topic by using Venn diagrams to illustrate some of the situations the students are exploring. Seeing the solutions in a number of different ways should help students understand the interconnectedness of these concepts. Most of the resource texts do not separate this concept completely from independent and dependent events, so the teacher should choose exercises with some caution.

Concept A: Probability

Objectives

A.3
To determine the probability of two or more independent events.

Instructional Notes

The students should be given some problem situation to solve using their knowledge of probability. This could be similar to finding the probability of rolling a five on a die and tossing a head with a coin. The students could do this experimentally, list the sample space, and read the result. The introduction of probability trees in some of these instances may help student understanding. After doing several of these types of questions, the students could be guided to the definition of independent events and the formula used to calculate the probability of independent events, that is, $P(A \cap B) = P(A) \bullet P(B)$

It is useful to introduce the definition of the complement of the probability of an event A at this point.

Note that: $P(A) + P(\bar{A}) = 1$

Examples/Activities

1. A bag contains 3 red marbles, 4 green marbles, and 7 blue marbles. Draw two marbles, one at a time, placing the first marble back in the bag before the second draw. What is the probability that we draw:
   a) a red, then a green;
   b) a red and a green;
   c) a red and a blue;
   d) a green and a blue;
   e) two red;
   f) two green;
   g) two blue?

   In three draws, always replacing the first marble, what is the probability of one red, one green, and one blue?

2. The probability that Jennifer will be chosen to run for her school's relay team is 2/3, while the probability that Ila will be chosen is 1/2, and for Kendra 2/5. Given that these events are independent, what is the probability that:
   a) Ila and Jennifer are chosen;
   b) Kendra and Jennifer are chosen;
   c) Ila and Kendra are chosen;
   d) Ila, Kendra, and Jennifer are chosen;
   e) Jennifer and Kendra are chosen, but Ila is not;
   f) Ila and Kendra are chosen, but Jennifer is not;
   g) Jennifer is chosen, but Ila and Kendra are not;
   h) none of them are chosen;
   i) at least one of the three is chosen;
   j) at least two of the three are chosen?

Adaptations

The teacher may adapt this concept by utilizing a more experimental approach in introducing this topic. Probability trees, or activities that involve manipulatives could be employed to reinforce the definition of independent events. It may also help the reinforcement of the concept to use a modified form of concept attainment, where students are given some examples of both independent and dependent events, and asked to categorize the two types.

Exercise 2 in the Activities column indicates some ways in which this concept can be extended. By introducing three or more independent events, and finding related real-world applications, better students can be challenged.
Students could be asked to solve problems 5 and 6 for Objective A.2 using another method.

Concept A: Probability

Objectives

A.4
To determine the probability of dependent events (conditional probabilities).

Instructional Notes

The definition of dependent events is crucial to a valid understanding of this concept. In order to help students attain this understanding, it might be useful to discuss real-world dependent events and their connectedness before doing any calculations.

Example. Using a deck of standard playing cards, discuss the effect of drawing two cards without replacing the first card drawn. Point out that it would be impossible to redraw the same card, that there would only be 51 (instead of 52) choices for the second draw, and so on.
Note: Many students are not familiar with playing cards, so a sample pack may have to be provided by the teacher or other students.

Have students work individually, in pairs or small groups on an exercise or two that allows them to work with dependent events. As an example, they might be given a paper bag containing three green and four red markers, and a series of exercises such as:

If we pull out two markers, one at a time, without replacing the first, what is the probability that we obtain two red ones? two green ones? one green and one red? (A probability tree may be utilized here.)

The students should eventually become familiar with the formula associated with dependent events.
$P(A \cap B) = P(A) \cdot P(B/A)$

Note that $P(B/A) = \frac{P(A \cap B)}{P(A)} = P(B)$
in the case where A and B are independent.

Examples/Activities

1. In their annual door prize draw, the employees of a store have one ticket entered for each year they have been with the company. Reid has been with the company for five years, Janet for 15 years, and Bill for 22 years. There are 258 tickets altogether. If winning tickets are discarded after being drawn, what are Reid's chances of winning the first two draws? Janet's chances of winning the first two draws? Bill's chances?

2. In drawing two cards from a standard deck of playing cards, one after the other, without replacement, what is the probability that:
   a) two clubs are drawn;
   b) two kings are drawn;
   c) an ace and a queen are drawn;
   d) two red cards are drawn; or,
   e) two face cards (jacks, queens, or kings) are drawn?

3. From a drawer containing six black socks, 8 white socks, and 4 red socks, Jon reaches in and selects two without looking. What is the probability that:
   a) he selects a pair of black socks;
   b) he selects a pair of red socks;
   c) he selects a pair of white socks; or,
   d) he selects a mismatched pair?

Adaptations

This topic can be extended by having students set up, but not necessarily solving, situations that involve a series of dependent events. For example, students might be asked the probability of being dealt six clubs in a row. Other similar exercises could be given.

The students could also relate their knowledge of combinations and permutations to the types of exercises above, in order to determine other types of valid solutions.

Probability trees can be utilized to illustrate the procedure involved in the solutions of these exercises.

Concept A: Probability

Objectives

A.5
To set up, analyze, estimate, and solve word problems based on objectives 1 - 4.

Instructional Notes

The students should be relatively familiar with the types of exercises posed in this section, but may encounter difficulty in deciding what type of situation is described in each exercise. To allow for such, some opportunity to practise identifying the types when they appear together should be provided. A series of statements representing the various types of exercises could be distributed. Students should identify, with justification, the concept involved. Once the underlying concept is correctly identified, then the students can be instructed to complete the calculations to find the indicated solution. (In some cases, it might be suggested that setting-up the correct solution is all that is required.)

Examples/Activities

Decide whether each of the following represents events which are mutually exclusive, independent, or dependent, and then complete the calculations to determine the indicated result.

1. What is the probability of being dealt two successive aces, if the first card is not replaced?

2. When rolling a die and tossing a coin, what is the probability of rolling a number greater than 4 or tossing a head?

3. What is the probability of drawing two hearts in succession, if the first card is replaced before the second is drawn?

4. What is the probability of rolling a 4 or a 3 on one roll of a die?

5. A student determines the probability of being accepted to the college of law is 3/5, to the college of engineering 1/2, and to the college of education 3/4. What is the probability that the student is:
   a) accepted by both law and engineering;
   b) accepted by engineering or education;
   c) accepted by all three;
   d) not accepted by any; or,
   e) accepted by law or engineering or education?

Adaptations

This section is intended to allow students to practise their newly-developed skills in a variety of situations, where they have to make a choice from several types of problems posed to them. The adaptation can occur in various ways, such as the level of difficulty or challenge in the exercises, or in the method and type of calculation required. A student might be instructed to estimate the probability before the calculation, asked to set up the calculation but not simplify, or be asked to use calculators or computers to provide the calculation.

Students who wish to be challenged may be given examples such as the following:

What is the probability of being dealt (without replacement) a five-card hand that includes three jacks and two tens?

See Appendix A for a discussion of Lotto 6/49 and related notations.

Objectives

A.6
To determine the coefficients of terms in a binomial expansion using the Binomial Theorem. (Pascal's Triangle or combinations could be used to introduce this topic).

Instructional Notes

This topic can be introduced in many ways, as indicated in the objective itself. The students should receive enough practice to become familiar with the coefficients of expansion and how to obtain them. It might be useful to actually expand a binomial of the type $(x + y)^4$ by multiplication, so that students actually see that the coefficients are in fact the same as those found using these other techniques.

The techniques most often utilized are Pascal's Triangle, combinations, or multiplication. (It is intended that students will use combinations by the end of this section.)

The teacher might have students complete the following exercise and relate the coefficients to Pascal's Triangle.

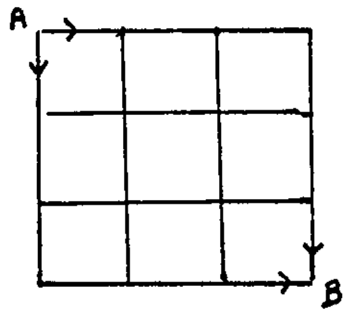
$$(x + y)^0 = 1$$
$$(x + y)^1 = 1x + 1y$$
$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3, \text{ and so on.}$$

Other activities are suggested in the next column.

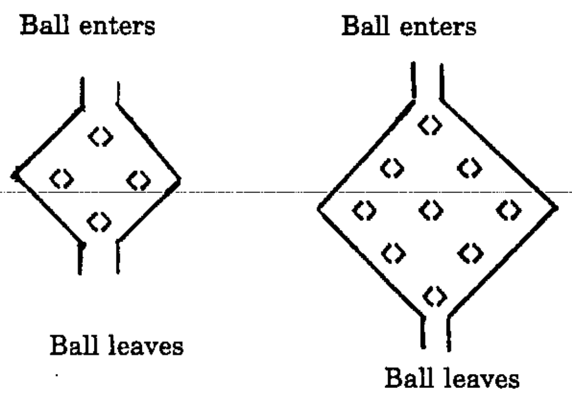
When students are familiar with these coefficients, they should be given some exercises enabling them to practise not only finding these coefficients but being able to determine the indicated term as well. The combinatorial approach should be emphasized.

Examples/Activities

Several types of activities can be suggested to generate the coefficients of a binomial expansion. One is the problem of determining how many routes there are from A to B in the following grid, if no 'doubling back' is allowed.



A second activity is to request students to design an elementary 'pinball' machine. In this machine, the ball is let in at the top, and must work its way to the bottom, without rebounding upwards. The students are to determine how many different paths the ball might take to reach the bottom. Two different pinball machines are drawn below.



1. Find the coefficient of the third term of $(x + y)^6$.

2. Find the coefficient of the middle term of $(b + g)^8$.

Adaptations

This topic could be adapted by asking students to determine the outcomes (listing, if necessary) of events which have approximately equal chances of occurring. As an example, students could be asked to determine in how many ways could there be a family of three girls and two boys. Similarly, if you toss a fair coin six times, in how many different patterns can you have two heads and four tails?

Students should also be expected to determine the coefficient and the proper sign of binomials whose initial coefficients are not equal to 1.
Use examples like:
a) the fifth term of $(2x + 3y)^7$ is ____

b) the third term of $(3x - 2y)^5$ is ____

c) the fourth term of $(2a^2 - 3xy^3)^6$ is ____

Students can also be asked to note the exponents of these terms and their relationships to each other and the original exponent.

Concept A: Probability

Objectives

Instructional Notes

A.7
To expand binomials of the form $(a + b)^n$ using the Binomial Theorem.

Students should be able to perform the expansion of common binomials using the Binomial Theorem. It is expected that students will be able to expand examples such as $(x + y)^6$, $(2x - y)^4$, and $(x^2 + 2yz)^5$ for test purposes, and will be able to expand more difficult binomials as part of their classroom exercises.

It is important to remember that while binomial expansion is a concept to be studied, many applications deal with a specific term of the expansion.

Most real-world expansions deal with two events that have approximately equal chances of occurring, such as the result of tossing coins, the appearance of boys or girls in family order, or the opening or closing of an electrical circuit. (TL)

When introducing the Binomial Theorem, it may be useful to have students expand some examples such as $(x + y)^5$ by multiplication and note the coefficients, the total of the coefficients, the number of terms, and the relationships of the exponents of each successive term. This may enable the students to understand the Binomial Theorem more easily.

A.8
To solve word problems associated with Objectives 6 and 7.

Most of the problems in this section deal with two events that have an equally likely chance of occurring, where several trials of the events are conducted. Students could work on these types of questions individually, in pairs, or in small groups, to practise the skills they have learned, not only in the last two sections but in the entire unit in the problem-solving context. (PSVS)

Examples/Activities

1. Expand $(x - y)^6$ using the Binomial Theorem.
2. Expand $(3x + y)^4$ using the Binomial Theorem.
3. Illustrate how many different ways a coin tossed seven times could land using the expansion of $(h + t)^7$. How many ways can we have less than three heads?

   Consider the special case where h and t are both equal to one.
4. Expand $(4x - 2yz)^5$.
5. Set up, but do not simplify, the expansion of $(2x - 5y)^9$.

Adaptations

These expansions could be done in several ways to deepen students' appreciation of the different approaches to mathematics. Introductory exercises could be done by multiplication as in the previous section, as well as by using the Binomial Theorem. These could also be introduced by completing one term at a time, in similar fashion to the preceding section.

See Appendix B for a related discussion and examples on the Binomial Theorem.

1. In a family of five children, in how many ways might one have three girls and two boys? What is the probability of having a family of three girls and two boys? Four girls and a boy? five boys? (Note that for our purposes, we are assuming that there is an equal chance of a boy or a girl being born, whereas in the real world there is, in Canada, a slightly higher than 50% occurrence of girls being born.)
2. If Asha's success rate in completing free throws is 50%, what is the probability that she will be successful on all four free throws she is awarded in a game?
3. In a series of tossing a coin, what is the probability that a fair coin will land 'heads' 8 times in a row ?
4. If a coin is tossed 12 times, in how many ways could four heads and eight tails appear? What is the probability of this occurring? In how many ways could six of each appear? What is the probability of this occurring?
5. In a family of seven children, what is the probability of having at least five girls? at least six girls? all but one boy?
6. What is the seventh term of the expansion of $(4x^3 - 2y)^9$?
7. In a true-false test of twelve questions, how many different ways can five true and seven false answers be arranged? How many different test solutions are possible?

Many of the types of problems associated with binomial coefficients are closely related to combinations and probability. Students may find it useful to refer to these sections when attempting to do problems of this type.

Students should be encouraged to use a variety of problem-solving techniques, including the listing of all possibilities, if they are experiencing difficulty in the solution of these problems.

Concept B: Data Analysis

Foundational Objectives

- To determine the standard deviation of a set of data and to utilize the standard deviation in analyzing that set of data (10 02 01). Supported by learning objectives 1 to 3.
- To develop skill in interpreting data through the use of z-scores (10 02 02). Supported by learning objectives 4 to 6.

Objectives

B.1
To describe and illustrate normal and skewed distributions using real-world examples.

Instructional Notes

This section is to be treated as an introduction to the concept of normal and skewed distributions. It is largely intended to allow the students to identify, analyze, and interpret these distributions. The objective of having students draw a normal distribution for a given set of data should be postponed until students have learned how to calculate standard deviations, which is done in objectives 2 and 3 of the unit.

In this objective, students should become familiar with the definitions of the terms involved, the general appearance of normal and skewed distributions and be able to interpret the information presented by a given diagram. Most of the newer recommended resources include sections on these topics. Basic definitions are included and illustrated.

The teacher might present some given information, and ask students to interpret and expand on the information, as well as instruct the students to identify those elements associated with the basic definitions.

Example.

The heights of 600 male secondary school students are normally distributed. The average height is determined to be 170 cm and the standard deviation is determined to be 6.5 cm. Draw a normal curve to illustrate this situation.

About how many of these students would be more than 183 cm tall or less than 163.5 cm? How many students would you expect to be between 163.5 cm and 176.5 cm tall?

Examples/Activities

Have students work individually, in pairs, or in small groups, on the following types of exercises.

1. The weight of beef roasts is normally distributed with a mean of 2.35 kg and a standard deviation of .5 kg. If a butcher cuts 350 roasts a day, what number of roasts would you expect to weigh more than 2.85 kg? What is the probability that you would choose one of these roasts at random? How many roasts would you expect to weigh less than 1.35 kg?

2. You are aware that the weights for children of age 10 are normally distributed with a mean of 40 kg and standard deviation 8 kg. As an assistant medical worker, you weigh a group of sixty 10 year olds and find their weights to average 31 kg, with a standard deviation of 4 kg. Illustrate both sets of data by drawing the distributions (bell curves) for each. What might some interpretations of these two sets of data conclude? Write your answer as a brief report to your supervisor.

3. A buyer for a department store chain is aware that the sizes of men's shoes for Canadian men are normally distributed with a mean of size 10 and a standard deviation of 1.5 sizes. For the chain, he purchases 12 000 pairs of shoes to fit this distribution. In this batch, how many pairs of shoes would be suitable for a man who has a shoe size greater than size 13? How many pairs would there be within one standard deviation from the mean?

Adaptations

Students could be provided with a set of bell curves where the mean and standard deviation are noted. They should be asked for their interpretations and to decide whether these would represent normal or skewed distributions. They could also be asked to write brief reports based on the details provided by the graphs, outlining their analysis and interpretation.

These graphs might be suggested by newspaper or magazine articles, found in other print sources, gathered from local officials, or assigned from recommended resources. (IL)

Concept B: Data Analysis

Objectives

B.2
To calculate the standard deviation of a set of data.

Note: There are two statistical standard deviations; one calculated for a sample

$$0 = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n-1}}$$

and one calculated for a population.

$$0 = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

The teacher might wish to make students aware that these two methods exist; but for purposes of this course, only the standard deviation for a population will be used (where the divisor is simply n).

Instructional Notes

Students should be introduced to the calculations required to determine the standard deviation for a given set of data. Once they understand how the standard deviation is to be determined, they should be provided data from which they can determine the mean and the standard deviation. The students should be able to make use of appropriate technology in doing the calculations. Students may also find it more valuable to work in pairs or small groups, in order to monitor each others' progress and calculations. (PSVS)

The data provided could be assigned from recommended resources, newspapers, or magazines, could be class-generated (e.g.: height, marks, height of standing jump, number of successful free throws in twenty-five tries), or from other subject areas. (IL)

The first calculations should be based on finite sets of data, small enough in number that the calculations themselves do not overshadow the main objective. As students become more familiar with the objective, the number of elements in the distribution can be increased.

As an additional exercise, you may wish to compare frequency distributions such as:

a) 100 elements all having a value of 50;
b) 50 elements having a value of 1, and 50 having a value of 99.

Note that both distributions have the same mean but their standard deviations are far from equal. This illustrates the need of more than one parameter, such as the mean, to describe a population.

Examples/Activities

1. Given the following scores obtained by local basketball teams in the last several games, determine the mean number of points and the standard deviation.

37	46	55	31	72	66	61	52
29	63	47	36	58	73	65	56
44	27	53	66	70	62	41	39
64	58	37	62	51	54	48	57
50	49	56	60	45	29	54	56

2. The following table represents a test of the number of hours that lightbulbs worked before burning out. These were selected at random from a shipment from a supplier. Determine the mean length of time the lightbulbs lasted and the standard deviation.

69	122	108	196	145	125
98	89	120	142	135	106
122	102	99	163	116	131

3. The following table represents the number of pairs of pants sold by a local mens' wear store in the last month. Determine the mean size, and the standard deviation.

Waist Size	Number of pairs sold
26	2
28	1
30	9
32	18
34	12
36	10
38	6
40	4
42	2
44	0
46	1

Adaptations

For each of the exercises in the previous column, the addition of a statement asking students to depict the information calculated in a histogram could be part of an assignment. Then the students could be asked whether the given values depict a normal distribution, and/or whether some of the given values would be the expected ones. If some of these values do not fit in a normal distribution, students could be asked to supply reasons for this.

Students might also be asked to discover why we use

$$\sqrt{\Sigma(x-\bar{x})^2}$$

rather than

$$\Sigma(x-\bar{x}).$$

Objectives

B.3
To utilize the standard deviation to interpret data represented by a normal distribution.

Instructional Notes

Given a few exercises involving means and standard deviation, students should be able to determine and justify results based upon these. Students could work in pairs, and use appropriate technology, to aid them in their calculations. (TL)

Wherever possible, exercises should reflect local conditions. If this is not possible, most of the recommended resource texts have an adequate supply of exercises on this topic. Other sources of available statistics are Stats Canada, monthly bank magazines, stock market quotes in most newspapers, and on databases via Internet, and the daily news of stock markets, including agricultural commodities, as reported on TV newscasts. (NUM)

Examples/Activities

1. Assuming the weights of fish are normally distributed, Fisheries Department officials have determined that fish in a remote northern Saskatchewan lake average 1.5 kg., with a standard deviation of .6 kg. Of 1 000 fish, how many should weigh from .3 kg to 2.7 kg? Would a fish of 4.5 kg be very common? How many fish of 4.5 kg or greater would we expect in this population?

2. Have students analyze a stock market report page from a local newspaper. Get them to determine the mean change in stock value for the first fifty listed stocks. Have them determine the standard deviation for their data set. Do these represent a normal distribution? How many of these stocks change by more than two standard deviations? How many would we expect to change by more than two standard deviations in a normal distribution?

3. Have students note the scores from all the weekend games reported in the local paper. Have them determine the mean score per team, and the standard deviation. Illustrate these results on a bell curve. Do these scores form a normal distribution? (Perhaps these stats can be compared to the listed selections of the lottery game Over/Under.)

4. Assuming the lifetimes of the batteries they produce are normally distributed, workers randomly carry out tests to determine if the batteries meet the requirements of lasting a mean time of 144 hours and having a standard deviation of 8 hours. In a shipment of 2 400 batteries, how many would you expect to last 160 hours or more? How many should last between 136 and 152 hours? If you had four batteries that died after 80 hours, and another six after 96 hours, would you conclude this was a defective batch?

Adaptations

As an extension of this topic, students could be assigned some specific set of data to research, asked to calculate the mean and standard deviation, and required to present to the class/teacher as a report. This type of data might include size of deliveries at the local elevator, size of farms in hectares, precipitation for the local area, temperature, wind speed, length of growing season, or number of vehicles passing through a specific location in five-minute intervals. (COM)

Objectives

B.4
To define and calculate z-scores.

Instructional Notes

In some text resources, z-scores are explained as values taken from a set whose mean is zero and whose standard deviation is one.

In others, the z-score is indicated to be a quantifier determined by subtracting the mean from each value, and dividing the difference by the standard deviation. Although the end result is the same, students may find it easier to understand the second version. The teacher should introduce the topic in this manner.

In this case, z is determined by

$$z = \frac{x - \bar{x}}{s}$$

where $\bar{x}$ is the mean and s the standard deviation.

Emphasize that z measures how many standard deviations x is from the mean.

Students can be given several types of exercises in which they are to determine the z-scores for sets of data. They could work on these individually, or in pairs, and utilize appropriate technology.

See the note under the Adaptations.

Examples/Activities

1. Calculate the z-scores for the following set of data:

34	42	52	43	37	57
40	35	48	54	47	50
44	51	47	52	46	48

2. The following are the number of successful free throws thrown by the girls' basketball team in a recent practice. Each was given twenty-five free throws and results for all fourteen girls were recorded. Calculate the z-score for each.

11	17	6	10	12	8	11
13	9	10	12	9	10	12

3. On a recent quiz, the twenty-two students of Mr. Campbell received the following scores out of a total of 57. Calculate the z-score for each student.

29	44	37	51	42	54	36	45
43	39	46	48	50	46	44	
38	48	24	51	47	49	52	

4. Have the students collect some data on their own and calculate the z-scores for that data. This could be in the form of temperatures, precipitation, weight of cattle offered for sale, number of tonnes of grain delivered to the local elevator, number of words on a page, and the like. (IL)

Adaptations

Students should understand the relationship between the calculation of the z-score and the normal distribution. The text resources that deal with z-scores also include a table indicating the area under a standard normal distribution curve. At this point, it might be useful to indicate to students how one reads these tables, both to find the area under the curve and to find the z-score when given the area.

Some exercises on reading these correct values can be given to the class.

Objectives

B.5
To be able to utilize z-scores as an aid in interpreting data.

Instructional Notes

Calculation of z-scores allows us to compare data obtained from different sets, which have different means and different standard deviations. Some examples in calculating and comparing z-scores can be done for the class, before any exercises or activities might be assigned.

For any value x in a set of data, the z-score, z, can be determined by

$$z = \frac{x-\bar{x}}{s}$$

where s is the standard deviation.

In this section, the teacher may wish to have students supply data of their own. This necessitates assigning this task in advance of the lesson itself. Students might be encouraged to begin collecting data early in the course, to have adequate entries to use. Alternatively, the teacher may wish to use teacher-selected data. Again, this should be prepared in advance. Statistics Canada is a good source of such data, databases on the Internet as well as local developments. Text resources also supply a good cross-section of exercises. (IL) (TL)

B.6
To solve related real-world problems using statistical inference.

The problems introduced to students in this section can be developed through student-generated data, locally-generated data, data supplied by the teacher or obtained from text and other resources. In most of these situations, it may be useful to have students work in pairs or in small groups, as they can double-check calculations, table values, and provide each other with immediate assurances or doubts about strategies. (PSVS)

These problems can also be constructed to lend themselves to brainstorming, and to Creative and Critical Thinking. The teacher may wish to have students do summary problems; that is, to introduce a problem that reviews all sections of this unit, or to do problems that have a specific focus. (CCT)

Examples/Activities

1. One activity that students might wish to do on their own is to compare their own marks on two tests to determine the z-scores. If Chandra obtained a 68 on a test where the class average was 62, with a standard deviation of 5 and obtained a 77 on a test where the class average was 72, with a standard deviation of 8, on which test did she obtain the higher z-score? Students could obtain their own mark information from the teacher and compare some of their results in this manner.

2. Find the z-scores for the weight of each player on the senior football team (volleyball team, basketball team, hockey team).

3. A fishing derby on a lake gives first place to the fisherman who obtains the best result for that lake for the day. Three species are quite commonly caught and it is assumed that the weights of the fish are normally distributed in the lake. Perch have a mean weight of .5 kg, with a standard deviation of .2 kg; pickerel have a mean weight of 1.4 kg, with a standard deviation of .6 kg; and northern pike have a mean weight of 2.3 kg, with a standard deviation of .8 kg. If Ali brings in a perch of .9 kg, Bob a pickerel of 2.5 kg, and Cheryl a northern pike of 3.9 kg, who should be awarded first prize?

1. Assuming a normal distribution for the life of its test batteries, a company has found that its batteries have a mean life of 72 hours, with a standard deviation of 9 hours. What are the z-scores of two batteries which last 68 and 84 hours respectively? What is the probability that if we choose a battery at random, we will obtain one that will last anywhere from 68 to 84 hours? What is the probability that we will choose a battery that lasts less than 64 hours?

2. On a standardized math test where the marks are normally distributed, the results show a mean of 73, with a standard deviation of 11. What is the probability of randomly selecting a student whose mark is more than 86, less than 68, between 50 and 70?

3. On which of the two following tests did Jean get the better result:

	Test A	Test B
Mean	68	73
Stan Dev	12	15
Jean's mark	75	82

4. If a well-known standardized IQ test has a mean of 100, with a standard deviation of 15, how many Canadians (of 30 000 000) would be expected to have IQs greater than 145, if the scores are normally distributed?

Adaptations

Students could be given sets of data from which to calculate the mean, standard deviation, and z-scores. This could be done in pairs.

Example.
Use the following sets of data to calculate the mean, standard deviation, and z- score for each element (number of free throws in twenty five tries).

Team A

6	9	7	8	10	15	7	11
9	14	12	11	13	10	11	7

Team B
(number of free throws in fifteen shots)

7	9	12	4	6	8	8
9	10	8	7	9	8	6

Which player would you expect to win the free throw competition at a game involving these two teams? Who would you expect to obtain the fewest free throws?

If the students and teacher are comfortable with access to class marks, heights, shoe sizes, and the like, this type of data can lend itself to comparative analysis quite easily. The advantage in using data such as class marks is that it may in itself serve as a motivator for some students, and in some small way as a diagnostic tool for others. Teachers should again remind students of the role of statistics and that, interpreted in various ways, they can lead to misleading information.

Questions or exercises on intelligence tests, learning ability tests, and physical characteristics can all be introduced, as many of the practical applications are based on human measurements. Use these only if the class feels comfortable with such measurements.

Concept C: Matrices

Foundational Objectives

- To illustrate appropriate real-world situations using matrices (10 03 01). Supported by learning objective 2.
- To demonstrate knowledge of terms associated with matrices (10 03 02). Supported by learning objectives 1 and 6.
- To develop skills in matrix operations and in solving related real-world problems (10 03 03). Supported by learning objectives 3, 4, 5, 7, 8, 9, 10, 11, and 12.

Objectives

C.1
To define basic terms associated with matrices.

Instructional Notes

Students should be made familiar with the definitions of matrix, rows, columns, entries, row matrix, column matrix, elements, dimensions of a matrix, square matrix, zero matrix, and other introductory terms. These definitions can be learned through identification practice. The teacher can write a matrix on the board and have class members identify various elements, rows, dimensions of the matrix.

The teacher may also wish to indicate to students some of the applications of matrices in the real world. These include aspects of inventory control, data management systems, traffic management systems, and linear programming. More specific examples of these types can be shown at this point, or introduced throughout this unit.

Students should be able to demonstrate their knowledge of the basic terms by correctly identifying them from sample matrices.

See Appendix C for an alternative method of introducing matrices and for some of their applications.

Examples/Activities

1. For the matrix

$$A=\begin{bmatrix} 6 & 4 & 0 & -2 \\ 1 & 3 & 5 & 1 \\ 2 & 6 & 1 & 0 \\ 3 & -5 & 2 & 7 \\ -5 & 2 & 4 & 1 \end{bmatrix}$$

   a) State the dimensions of the matrix.
   b) How many elements are in the matrix?
   c) List the elements of the third row.
   d) List the elements of the fourth column.
   e) What element is in the third column and the second row?
   f) The element in the fifth row and first column is ____.

2. Form a matrix that has dimensions of 3x2. List the element in the second row and first column.

3. Create a row matrix having three elements. What are the dimensions of this matrix?

4. Create a column matrix of five elements. What are the dimensions of this matrix?

5. Create a square matrix of nine elements. What are the dimensions of this matrix?

6. In the matrices given below, find the value of each variable which will make the matrices equal.

$$A=\begin{bmatrix} 4 & x & -2 \\ y & 1 & z \\ 6 & w & v \end{bmatrix} \quad B=\begin{bmatrix} a & -4 & c \\ 5 & d & 3 \\ f & 0 & 8 \end{bmatrix}$$

Adaptations

Students might be given exercises such as the following.

What values of x and y would make matrix A and matrix B equal?

$$A=\begin{bmatrix} 3x+2y \\ 2x-4y \end{bmatrix} \quad B=\begin{bmatrix} 2 \\ -20 \end{bmatrix}$$

Objectives

C.2
To create a matrix to illustrate a given situation.

Instructional Notes

Students can work individually, in pairs, or in small groups, to create matrices for situations described and outlined by the teacher, or found in resource texts. An example or two done for the entire class might be a useful starting point. Use the vocabulary developed in the first section as reinforcement of these terms. (PSVS)

Some examples from which matrices are created can be found in the next column, however, the teacher may wish to use others that have a more local flavour.

Draw attention to the creation of matrices in Spreadsheet and Word Processing programs the students might have studied.

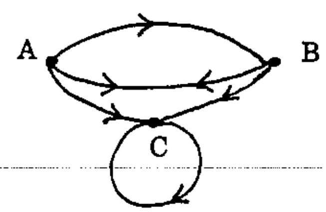
Examples/Activities

1. The Platters' music store sells records, cassettes, and compact discs. Create a matrix illustrating the following sales:

C&W	31 records	128 cassettes	63 CDs
Rock	42 records	85 cassettes	38 CDS
R&B	27 records	64 cassettes	55 CDs

2. Food Wholesalers' warehouse supplies vegetables (in kg) to four food chains in the area. To All-Rite Foods, they sell 60 kg of peas, 257 kg of potatoes, and 72 kg of carrots. To Mainly-Fine Foods, they sell 45 kg of peas, 187 kg of potatoes, and 59 kg of carrots. To Grate Foods, they sell 73 kg of peas, 225 kg of potatoes, and 82 kg of carrots. To Hardly Foods, they sell 65 kg of peas, 168 kg of potatoes, and 62 kg of carrots. Create a matrix to illustrate the above sales.

3. For the following schematic road map, create a matrix that illustrates the number of ways to travel between A, B, and C, where arrows indicate the directions to be followed, and routes must be one-step (not through any connecting point). Then create a two-step matrix based on the diagram.



One-step solution

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Adaptations

Once students have had a little experience with matrices, it might be interesting for them to use matrices to model some situations in their personal domain. These could reflect the number of routes to and from a few of their most common destinations (school, a friend's, a confectionery). Other suggestions might include varieties and quantities of grain delivered to their local elevator, sales from the local co-op store, garage, or ranch. (NUM)

Concept C: Matrices

Objectives

C.3
To add and subtract matrices.

Instructional Notes

Students could be asked to create matrices based on a given situation and then be instructed to find the sum of the matrices based on this information. This could be done by posing a problem and allowing students to work toward the solution individually, in pairs, or in small groups. Students should be instructed to use matrices in their solution of the problem, and to justify their solutions. (PSVS)

By comparing the work of various students or groups, it may be possible to illustrate the commutativity of addition of matrices. When students have successfully completed the problem, more exercises can be assigned. In the case of numerical exercises, students might be asked to create a hypothetical situation that exemplifies the matrices to be added, and their sum.

Ask students to state any conditions that they feel must be met before matrices can be added or subtracted (e.g.: matrices must have the same dimensions before they can be added or subtracted).

Examples/Activities

1. A musical supplier, Hi-Note Inc., supplies records(R), cassettes(C), and compact discs(CD) to three music stores (Do, Re, and Mi) in the local area.

   In February, the number of records sent out were 47 to Do, 62 to Re, and 55 to Mi. The cassettes were 123 to Do, 145 to Re, and 172 to Mi. The CDs were 73 to Do, 82 to Re, and 66 to Mi.

   In March, the records were 72 to Do, 68 to Re, and 71 to Mi. Cassettes were 153 to Do, 149 to Re, and 193 to Mi. CDs were 82 to Do, 79 to Re, and 53 to Mi.

   Create a sales matrix for each month, and then add the two matrices to illustrate the total for a two-month period. Label rows and columns so you can justify your solution.

2. Add.

   a)

$$\begin{bmatrix} 6 & -3 & 5 \\ 2 & 5 & -1 \\ -3 & 7 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 & -1 \\ 0 & -5 & 5 \\ 3 & -2 & 8 \end{bmatrix} =$$

   b)

$$If\ A = \begin{bmatrix} 4 & 7 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

   and

$$B = \begin{bmatrix} 3 & -5 & 4 \\ -5 & 3 & 6 \end{bmatrix}$$

   Find A + B, A - B, B + A, B - A.

3. Solve for the matrix X.

$$X + \begin{bmatrix} 4 & -3 & 6 \\ 0 & 2 & -4 \\ -5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -1 \\ -3 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

Adaptations

This topic might be adapted by introducing variables as missing elements and having students determine the value of each of the variables. This would necessitate the understanding of matrix addition on the part of the student and would provide a little practice in solving simple equations.

Example.

Find the value(s) of each of the variables in the following addition.

$$\begin{bmatrix} x & 3 \\ 4 & 2y \\ -2 & 2z \end{bmatrix} + \begin{bmatrix} 2 & g \\ -2 & z \\ 7 & y \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ h & 5 \\ j & 1 \end{bmatrix}$$

Concept C: Matrices

Objectives

C.4
To add and subtract matrices using scalar multiplication.

Instructional Notes

Students can be given a problem situation to work upon individually, in pairs, or in small groups, to determine their solution to the problem. They should be instructed to utilize matrices in their solutions and to justify their solutions. As well, they should be expected to be able to explain how to multiply a matrix by a scalar.

Examples/Activities

1.

a) The Sure-Shot Hockey Stick Co. supplies a sporting goods store with a monthly order of junior and senior hockey sticks. Each division comes in three styles: left, right, and goaltender. A monthly order for junior sizes is 98 left, 124 right, and 15 goaltender, while the senior order is 225 left, 283 right, and 28 goaltender. Use a matrix to illustrate these sales. The stick company decides to change to three-month ordering to facilitate book-keeping. Use a scalar multiplier to illustrate the matrix created under the new system.

b) The stick company also supplies a second store with a monthly junior order of 125 left, 167 right, and 23 goaltender, while the seniors are 289 left, 302 right, and 36 goaltender. Create a matrix of the same order as a), to illustrate the sales to the second store. The second store has purchased a record-keeping system that requires them to control inventory with a four-month total. Use a scalar multiplier to illustrate this four-month matrix.

c) The stick company wishes to determine a matrix that sums up its sales to the two stores in a) and b), for the respective periods of three and four months. Write out the matrix statement that illustrates this situation and determine the solution matrix.

2.

$$If\ A = \begin{bmatrix} 2 & -3 & 5 \\ -4 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} -4 & 2 & 0 \\ 2 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

Calculate:

a) 2A + 3B b) A - 2B c) 4A - B

d) 5A + 2B e) 3B - 2A f) -2A + 3B

g) -2A - 3B h) -4B + 3A i) A - -2B

Adaptations

This topic could be adapted by again using variables in the matrices for which students would be expected to determine the values of the variables that would make the statement true.

Example 1.

$$3X + 2\begin{bmatrix} 1 & 3 & -2 \\ 0 & 2 & -4 \\ 4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 12 & 4 \\ 12 & 13 & -14 \\ 11 & 2 & -10 \end{bmatrix}$$

Example 2.

$$2\begin{bmatrix} 4 & x & y \\ z & 2 & -3 \end{bmatrix} + 3\begin{bmatrix} a & y & x \\ -1 & c & d \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 4 & 1 & -12 \end{bmatrix}$$

Concept C: Matrices

Objectives

C.5
To multiply two matrices (not larger than 3x3).

Instructional Notes

Students should work individually, in pairs or in small groups, in order to provide immediate feedback and to check calculations in each others' work. The teacher should illustrate the process of matrix multiplication, starting with a row matrix times a column matrix, then a matrix times a column matrix, and so on. The teacher should outline or illustrate the process and have students determine which types of matrices can be multiplied.

Students should be given both multiplication exercises and real-world problems to practise multiplication.

If students do not determine the fact that matrices can be multiplied only when the number of rows in the first matrix is equal to the number of columns in the second matrix, (A_{mxn}, B_{nxp}, with the result being C_{mxp}), then the teacher should be prepared to give examples or counterexamples that lead to this conclusion.

Students could also be given the opportunity to justify their solutions to others, especially if they are working with real-world situations. Many of the newer text resources have good real-world applications of matrix multiplication. (CCT)

Examples/Activities

1. a) The Whiz-Bang Fireworks Co. sells three different packages of fireworks. Package A includes 12 ladyfingers, 6 Tom Thumbs, and 3 sparklers. Package B includes 18 ladyfingers, 4 Tom Thumbs, and 5 sparklers. Package C contains 9 ladyfingers, 10 Tom Thumbs, and 8 sparklers. Create a matrix to illustrate these packages.

   b) The Company charges $.15 for each ladyfinger, $1.30 for each Tom Thumb, and $.85 for each sparkler. Create a matrix to illustrate these prices.

   c) Write a matrix multiplication statement that would allow the company to determine how much to charge for each of the packages of fireworks described in a). Complete the multiplication and show the price for each package in a matrix.

   d) The Big Bang Company sells the same three packages but they charge $.12 for ladyfingers, $1.35 for Tom Thumbs, and $.82 for sparklers. Create a matrix multiplication statement that compares both companies and their package prices. Do the indicated multiplication and identify the company that offers the best price to the consumer for each package.

2. Multiply

$$\begin{bmatrix} 6 & 1 & -2 \\ -2 & 0 & 3 \end{bmatrix} x \begin{bmatrix} 2 & -3 \\ 1 & 5 \\ 0 & -2 \end{bmatrix}$$

What are the dimensions of the product matrix?

Adaptations

This topic can be adapted by extending multiplication to three matrices. Students would have to determine if the multiplication could take place by checking the dimensions of the matrices and then carry out the required multiplications. It is also possible to have the students test to see if the order of multiplication is important. This will preview the next section.

Example.

Multiply.

$$\begin{bmatrix} 2 & 3 & -4 \\ 0 & -2 & 1 \end{bmatrix} x \begin{bmatrix} -3 & 1 \\ 2 & -3 \\ 4 & 0 \end{bmatrix} x \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Objectives

C.6
To determine the properties of matrices with respect to addition, scalar multiplication, and multiplication.

Instructional Notes

While students should be relatively familiar with the terms commutative, identity element, and zero element, these and others may have to briefly be reviewed in the context of integers or rational numbers. If the definitions are not familiar, the operations will be. When the students have reviewed these basic properties, they can be asked to work individually, in pairs or in small groups to determine analogous rules and elements for matrices. They can be instructed to create their own matrices to determine if addition is commutative, associative, distributive with respect to addition or subtraction, if there is an identity matrix for matrix addition, if there is an inverse for matrix addition, and the like. They should also be reminded that one correct counter-example is sufficient to negate the property. (CCT)

The students can then be asked to complete a similar activity for subtraction, then for scalar multiplication, and for matrix multiplication.

Students should also prepare a written summary for themselves after completing this activity. This summary should indicate which properties are valid for matrices and which are not. Included examples will make it easier for students to review their notes. (COM)

Examples/Activities

1. Is it possible to add two matrices whose dimensions are 2x3, or to multiply them?

2. Create three 3x2 matrices. Label them A, B, and C.
   a) Add A + B. Add B + A. Do you obtain the same result? Try B + C. C + B. Result? What property does this illustrate?
   b) Add (A + B) + C. Now add A + (B + C). Is the result the same? What property does this illustrate?
   c) Can you find a matrix X such that A + X = X + A = A?
   d) Is there a matrix Y such that B + Y = Y + B = Zero matrix?
   e) Does n(A + C) = nA + nC?

2. Create three matrices with dimensions 2x2. Label them A, B, and C.
   a) Does A x B = B x A? Does A x C = C x A?
   b) Does (A x B) x C = A x (B x C)?
   c) Does A(B + C) = AxB + AxC?
   d) Is there a matrix T such that T x A = A x T = A? Is this also true for T x B = B x T = B?

Adaptations

The teacher may adapt this topic by listing the properties which are associated with matrices and those which are not; rather than have students determine them on their own. In cases where the properties do not hold true, a counter-example should be given to illustrate why this property is not valid. The teacher may also wish to combine the two, having students determine some of the properties on their own, and to list or do others for the class as examples.

Objectives

C.7
To use row operations with matrices.

Instructional Notes

Row operations are skills that the students will employ in solving matrix equations in Objective C.9. As skills, they could be demonstrated to the students by the teacher and then practised on a few exercises that will help students attain these skills.

There are three types of row operations on matrices. Each of these is called an elementary row operation. All row operations are compositions of those elementary operations. The three types of elementary row operations are:

1. Interchange any row with any other row.
2. Multiply each element of any row by a non-zero scalar and then replace the original row by the new row obtained by this scalar multiplication. The new row is called a scalar multiple of the original row.
3. Replace any row by the sum of that row and a scalar multiple of some other row.

Students can obtain practice developing these skills. The first row operation can be shown by real-world example quite easily at this stage.

Examples/Activities

1. In the matrix

	Records	Cassettes	CDs
StoreA	78	135	82
StoreB	68	144	72

Would changing the order of the stores make a difference? Rewrite this matrix showing such a change.

2. In the matrix

$$\begin{bmatrix} 6 & 4 & 1 \\ -3 & 4 & 2 \\ 5 & -3 & 6 \end{bmatrix},$$

multiply the given row by a scalar that will produce the intended result for each of the following. Begin with the original matrix in each case.

a) First row, second element to be 1
b) First row, third element to be -5.
c) Third row, second element to be 4.
d) Second row, first element to be 6.

Write the matrix created by each operation.

3. For the matrix

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & 2 \\ 1 & 0 & -3 \\ 6 & 2 & 10 \end{bmatrix},$$

complete each of the indicated row operations. Show each result. Begin each question with the original matrix.

a) Multiply 1st row by 2, add to 3rd row.
b) Multiply 2nd row by -2, add to 4th row.
c) Multiply 3rd row by 1, add to 2nd row.

4. In the matrix

$$\begin{bmatrix} 3 & -2 & 4 \\ -5 & 3 & 1 \end{bmatrix},$$

multiply the second row by n and add the result to the first row, such that the third element of the first row becomes 0.

Adaptations

These row operations are skills that all students at this level should attain. If the teacher wishes, the practice exercises could include beginning with a given matrix and changing it through a series of row operations, instead of always working with the original matrix. As an example, in the matrix in exercise 2 in the previous column, the students could be instructed to carry out successive row operations to make the first element of row 2, and the first and second elements of row 3 equal to 0. This would allow the students to obtain a foretaste of the techniques needed in solving simultaneous equations using matrices.

Objectives

C.8
To determine the inverse of a "2x2" matrix.

Instructional Notes

Students should be expected to state the meaning of an inverse before beginning this topic. Once the meaning of an inverse has been reviewed, the teacher might ask the students to attempt to determine the inverse of a 2x2 matrix such as

$$A = \begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$$

The students should be reminded that by definition of inverse, the inverse A^{-1} must satisfy $AxA^{-1} = A^{-1}xA = I$, the identity matrix.

If students are not able to find the solution in a few minutes, the teacher should indicate the procedure to be used.
For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

first multiply ad, then bc and take the difference. This is the determinant, D. Then the inverse, A^{-1}, is determined by

$$\frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Students can then be given practice finding the determinant for 2x2 matrices and then be asked to determine the inverse of several 2x2 matrices.

Examples/Activities

1. Find the determinant of each of the following matrices.

   a) $\begin{bmatrix} 6 & -3 \\ 5 & 1 \end{bmatrix}$

   b) $\begin{bmatrix} 7 & 2 \\ -4 & 3 \end{bmatrix}$

   c) $\begin{bmatrix} 4 & 3 \\ -2 & 3 \end{bmatrix}$

   d) $\begin{bmatrix} -6 & 2 \\ 0 & 3 \end{bmatrix}$

   e) $\begin{bmatrix} -5 & 2 \\ 4 & 1 \end{bmatrix}$

2. Determine the multiplicative inverse of each of the matrices in exercise 1.

3. Determine the inverse of

$$\begin{bmatrix} -4 & 3 \\ -8 & 6 \end{bmatrix}$$

Can inverses be found for all matrices?

Adaptations

This topic may be adapted by introducing the determinant of a 3x3 matrix. Resource texts have adequate numbers of exercises on this extension. This may be introduced either through the use of diagonals, or expansion by minors.

Objectives

C.9
To solve matrix equations using multiplication by an inverse. (Orders higher than 2x2 could be solved using technology.)

Instructional Notes

Students could be presented with a set of two simultaneous equations in two variables and be asked to solve these using methods with which they are already familiar. Then they could be asked to create a matrix based on the coefficients and constant terms. If this is done successfully, have the students discuss as a class, in pairs or in small groups how they might transform the matrix to determine the solutions of the variables in the equations.

If some students are successful in this step, they might be asked to present their technique to the class for discussion.

If students are not successful, the teacher should indicate some possible methods of solution.

Alternatively, the teacher can model the two methods of solving equations based on previous objectives. For the system of equations $ax + by = m$
and $cx + dy = n$
Method 1. Row operations.

Create matrix.

$$\begin{bmatrix} a & b & m \\ c & d & n \end{bmatrix}$$

Use row operations to reduce a to 1, and c to 0, then the equations can be solved easily by substitution. (Or you may continue row operations to reduce d to 1, and b to 0 as well.)

Method 2. Using inverses.

For the same system as Method 1, create the matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ n \end{bmatrix}$$

Left-multiply both sides by the inverse. (Why?)

$$\frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \times \begin{bmatrix} m \\ n \end{bmatrix}$$

Simplify.

Examples/Activities

1. Solve, using the method of row operations.

   a) $3x + y = 7$, $2x - 5y = -1$ b) $4x + 5y = 8$, $6x - 4y = 11$

2. Solve, using the method of inverses.

   a) $3x - 4y = -10$, $2x + 5y = 1$ b) $x - 2y = 7$, $3x + y = -14$

3. A manager of a double movie theatre is preparing her day's admissions. However, the till had a malfunction and did not register all the information. She knows the total receipts from Theatre 1 are $905, and that there were 62 adults and 84 children in attendance. Theatre 2's proceeds are $1203, with 114 adults and 52 children. She needs to determine the admission charged adults and children on that day. She thinks for a moment and then proceeds to find the solution. Re-create her solution by writing equations to represent the given data and solve by using matrices. (CCT)

Adaptations

This topic can be adapted by having students solve for matrix equations of three variables x, y, and z. This can be done by inverses, or by row operations. Row operations may be found to be more suitable in this situation. Matrices of the order 3x3 can be handled quite quickly, if substitution is used once the entries under the left diagonal are made equal to 0. E.g.:

x y z

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{bmatrix}$$

Then the 3rd row is $0x + 0y + hz = i$ or $hz = i$.
The second row is $0x + ey + fz = g$ or $ey + fz = g$ and so on.

Then $hz = i$. Solve for z.

Then, $ey + fz = g$
(Substitute for z, and solve for y.)
Finally, $ax + by + cz = d$
(Substitute for y and z, and solve for x.)
You may also take the matrix all the way to:

$$\begin{bmatrix} 1 & 0 & 0 & m \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & p \end{bmatrix}$$

and we immediately see x=m, y=n, and z=p.

Questions of four equations in four variables, and five equations in five variables, etc., can be solved using calculators or computers, and might be assigned to students who like to use technology, or who might like to attempt these. (TL)

Objectives	Instructional Notes
C.10 To graph systems of inequalities.	The topic of inequalities has not been dealt with directly since the solution of inequalities in Mathematics 10. It may therefore be necessary to briefly review some of the aspects of solving inequalities. When introducing the graphing of inequalities on a coordinate plane, the differences between graphs containing <, >, ≤, ≥, should be specified. The teacher may find that some students will understand this topic more quickly if they begin with exercises such as $x > 2$, $y \leq -3$, before introducing other types of linear inequalities. These types of examples lend themselves to easily understanding the use of a test point to determine the region to be shaded. After students have been introduced to graphic inequalities, then other types of linear inequalities and systems of inequalities could be introduced. Students should be expected to understand and explain the regions which represent the intersection of the two solutions and those regions which represent the union of the two solutions. When students understand the procedure involved in the graphic solution of systems of inequalities, they might also be encouraged to solve these systems using graphic calculators, or appropriate computer software. (TL)
C.11 To determine the points of intersection of lines drawn in Objective C.10.	Points of intersection of a system of linear inequalities can be found by using the methods developed for solving systems of equations as done in C.9 or Mathematics A 30. The students should be instructed to locate an intersecting point, identify which inequalities are involved, and use these inequalities to determine the coordinates of the point of intersection. The inequality symbols can be replaced with equal signs for the purpose of determining the point of intersection. Then the students can briefly review the various types of methods available to them for solution and complete the exercise. Students could use calculators or computer software to help with calculations or graphing in this section. (TL)

Examples/Activities

Graph each of the following inequalities:

1. $x \geq 4$
2. $y < 3$
3. $y \geq 2x - 1$
4. $3x + 2y \leq 6$
5. $-5x + 4y \geq 10$

Graph each of the following systems of inequalities, highlighting the intersection of the inequalities.

1. $x \geq 0$
   $y \geq 0$
   $3x + 2y \leq 6$

2. $2x - 5y < 10$
   $3x + 4y \leq 12$
   $x \geq 0$
   $y \geq 0$

3. $x - 4y < -4$
   $3x + y > 9$

4. $2x + 3y \geq 8$
   $4x - 3y < 9$

5. Compare the regions obtained in exercises 1 and 2 to those obtained in exercises 3 and 4. How many linear inequalities do we need to completely enclose a region?

Adaptations

As an extension to this topic of graphing of systems of linear inequalities, students might be asked to graph some absolute value inequalities, such as $y \leq |x|$, $y \geq |x| + 3$, or $y \leq |x - 2| + 3$.

As well, students could graph regions formed by three inequalities such as
$x + 2y < 6$
$3x - y > 6$
$y \geq \frac{1}{2}x - 8$

Exercises involving a system of four, five or more linear inequalities should also be introduced for student practice and discussion of results.

Determine all points of intersection for each of the following systems of inequalities. Students may find it useful to graph each system and note the coordinates at the point of intersection when they have been determined.

1. $3x - y \geq 2$
   $4x + y < 8$

2. $x + 3 > 0$
   $x + y \leq 6$
   $2x - y < 6$

3. $y > 4x - 3$
   $-3x + y \leq 4$
   $-x + 2y > -12$
   $2x + y \leq 3$
   $2x + y \geq -12$

Several other examples could be done, including real-world problem situations. These are available from the resource texts which include the solution of systems of linear inequalities. (IL)

Students might be instructed to graph these systems on a graphic calculator, or a computer, and then utilize the various trace and zoom features to identify the coordinates of the points of intersection. The degree of accuracy required could be determined through discussion in the class itself.

Objectives

C.12
To determine which vertices of the polygon formed by a system of inequalities maximizes or minimizes a given linear function.

Instructional Notes

Students should become familiar with the words or phrases that indicate an optimization is required. Students could be asked to generate a list of words such as maximum, minimum, greatest, least, highest, lowest, largest, smallest, and so on.

In this section, the focus should be on drawing polygons from information given in real-world problem situations and using this to determine the solution to a question written as an optimization.

Not all the text resources cover this topic, so the teacher may have to utilize alternate sources, or prepare exercises and activities in advance.

The students may be instructed to determine what optimization is required and write an equation based on this optimization. The other factors which affect this optimization must be included and equations or inequalities written. When all factors are taken into account, the statements are graphed and the vertices determined. The coordinates of the vertices can then be substituted into the optimization equation to determine which one represents the solution. This is a simple example of linear programming.

Examples/Activities

1. A recent math contest for high school students had questions categorized in sections A and B. Questions in section A took four minutes to solve, and were worth 6 points each. Those from section B took 6 minutes each, and were worth 10 points each. You are to answer a total of 12 questions, and are given 60 minutes to complete this test. How many questions of each type should you answer, in order to obtain the best possible score, if your answers are all correct?

   Solution:
   Optimization equation. For 'best score' (T)
   $T = 6x + 10y$
   Other factors.
   Time to write test: $4x + 6y \le 60$
   Number of questions: $x + y \le 12$
   Section A questions: $x \ge 0$
   Section B questions: $y \ge 0$

   Graph those inequalities representing 'other factors'. Points of intersection are (0,0), (0,10), (12,0), (6,6). Substitute each of these into the optimization equation ($T = 6x + 10y$) to determine the solution that gives the best score.
   Answer: (6,6)

   Note (0,0) is the minimum score.

Adaptations

Similar types of real-world situations could be assigned classes to solve. These could have parameters that are not necessarily the x - and/or the y - axes.
Example.

Victoria wants to invest up to $30 000 in Eagle or in SkyHi mutual funds. She wants to invest at least $13 000, but not more than $23 000, in Eagle funds and no more than $20 000 in SkyHi funds. Eagle funds offers a rate of return of 6% per year, while SkyHi funds offers a rate of 7% per year. How much should she invest in each to maximize her income? What is the maximum income? (NUM)

Concept D: Complex Numbers

Foundational Objective:

- To demonstrate the skills developed in operations with complex numbers (10 04 01). Supported by the following learning objectives.

Note: The teacher may combine Concept D: Complex Numbers and Concept E: Quadratic Equations in order to form one larger unit of study.

Objectives

D.1
To define and illustrate complex numbers.

Instructional Notes

The teacher may wish to complete a very brief review of the number system at this point, illustrating the numbers belonging to the various sets, and also noting why some numbers are real numbers and others are not. As this is the first instance of dealing formally with complex numbers, definitions are important. The concept of $i^2 = -1$, and therefore, $i = \sqrt{-1}$ should be introduced.

The teacher may wish to have the students practise some exercises using this concept, in order for it to become familiar to them.

To simplify expressions involving powers of i, the teacher can introduce the values of i, i^2, i^3, and i^4 , and develop the strategy of using $i^4 = 1$ to simplify.

Examples/Activities

Adaptations

1. Simplify each of the following expressions.

   a) $4i^2$

   b) $-3i^3$

   c) $2i^5$

   d) $6i^4$

   e) $7i^9$

2. Find the simplest form of the expression i^{93}.

3. Express in terms of i.

   a) $\sqrt{-5}$

   b) $\sqrt{-10}$

   c) $\sqrt{-48}$

4. Simplify.

   a) $\sqrt{-3} \cdot 4i$

   b) $\sqrt{-6} \cdot 5i^3$

   c) $15 \cdot 6i$

   d) $4i \cdot 7i$

   e) $\sqrt{-6} \cdot \sqrt{-3}$

Concept D: Complex Numbers

Objectives	Instructional Notes
D.2 To express complex numbers in the form a + b*i*.	Students could be shown that both real and imaginary components are part of a complex number. The complex numbers written in this form can also be illustrated on an Argand-Gauss coordinate plane, where the x-axis is the real part of the complex number, a, and the y- axis represents the coefficient of the imaginary component, b. The teacher may also wish to indicate to students the definition of the absolute value of a complex number at this time, using the plotted number on the coordinate plane and the Pythagorean Theorem.
D.3 To add and subtract complex numbers.	The introduction to addition and subtraction of complex numbers can be done in a number of ways. The teacher may wish to ask students to decide how to add two numbers such as $(4 + 3i)$ and $(3 + 5i)$, reminding them of the real and imaginary parts of complex numbers. Most classes will correctly add the reals to the reals and the imaginaries to the imaginaries. Alternatively, the teacher may outline the algorithmic procedure, present examples, and have students practise this procedure. A third method is to present this topic graphically. By treating each complex number as a vector and using the origin as a starting point, students can demonstrate the addition of the two complex numbers above by starting at the origin and plotting $(4 + 3i)$. Then using $(4 + 3i)$ as a starting point, Plot $(3 + 5i)$ by moving three units to the right and five units up. The total of the two complex numbers is represented by the coordinates of the last point.

Examples/Activities

1. Plot each of the following on a coordinate plane. Find $|a + bi|$ for each as well.

   a) $3 + 4i$
   b) $-5 + i$
   c) $-3 - 2i$
   d) $7 + 4i$
   e) $6 - 8i$

2. Write as a complex number.

   a) $4 - \sqrt{-2}$
   b) $\sqrt{-6} + 5$
   c) $\sqrt{-7}$

3. Simplify.

   a) $4(2 + 5i)$
   b) $6(5 - 3i)$
   c) $-6(4 - \sqrt{-3})$
   d) $8i^3 \cdot -3i^5$

1. Complete the indicated operation.

   a) $(4 + 7i) + (3 + 3i)$
   b) $(6 - 3i) + (-9 + 2i)$
   c) $(-8 + 5i) + (6 - 3i)$
   d) $(2 - 5i) - (-3 + 4i)$
   e) $3(2 - 3i) + 4(6 + i)$
   f) $2(4 - 3i) - 3(5 - 2i)$

2. Illustrate the following operations on a graph. State your solution.

   a) $(3 + 2i) + (4 + 3i)$
   b) $(3 - 4i) + (2 + 2i)$
   c) $(-2 - 3i) - (3 - 2i)$
   d) $2(3 + i)$

Adaptations

An extension to this objective and Objective D.1 is to assign students the task of researching the historical development and use of complex numbers. This can be done by reading the introduction to complex numbers in several different mathematics text resources, or by allowing students to use their skills in locating information from the learning resource centre. (IL)

Another source is NCTM's *Historical topics in the mathematics classroom.*

This information can be written in a paragraph or two. It may help some students appreciate the value of complex numbers.

The teacher might have students determine if the properties that are associated with addition and subtraction for real numbers are also true for complex numbers. That is, are commutativity and associativity valid for complex numbers? What about the distributive property of multiplication with respect to addition? Students might also be expected to determine the identity element for addition and the additive inverse for complex numbers. Are they the same as for real numbers? (CCT)

Concept D: Complex Numbers

Objectives	Instructional Notes
D.4 To multiply and divide complex numbers.	The teacher may wish to begin with multiplication of complex numbers and then proceed to division. Multiplication of complex numbers can be introduced as an algorithmic procedure, or the students may be given some exercises to attempt to complete on their own, or in small groups. Before moving to division, the teacher may find it helpful to review the concept of rationalizing a denominator.
D.5 To divide complex numbers using conjugates.	Students should be reminded of the process of rationalizing the denominator and then could be asked to determine what multiplier is to be used when attempting to rationalize a complex number such as $(4 + 3i)$. Students might be given a series of multiplication exercises that lead to the development of this concept. Example. $\sqrt{3}\ \sqrt{3} =$ $\sqrt{-5}\ \sqrt{-5} =$ $6i \cdot 6i =$ $(a + b)(a - b) =$ $(4 - \sqrt{5})(4 + \sqrt{5}) =$ Students should be reminded of the term conjugate and be expected to use it in describing the process.

Examples/Activities

1. Multiply.
   a) $6(3 + 2i)$
   b) $5i(4 - 3i)$
   c) $(3 + 2i)(5 - 3i)$
   d) $(5 - 3i)(-2 + 7i)$

2. What is the square of $(2 + 3i)$; the cube; the fourth power?

3. Simplify.

   a) $5/i$
   b) $7 \div (3i)$
   c) $-12 \div (4i)$
   d) $(6 - 5i) \div (2i)$
   e) $\frac{(3 + 2i)(5 - 3i)}{7i}$

4. Write the reciprocal of each of the following. Simplify.

   a) $3i$
   b) $\sqrt{-8}$
   c) $9i^3$

1. Complete the division.
   a) $5 \div (3 + 2i)$
   b) $-12 \div (5 - 3i)$
   c) $(4 + i) \div (1 + i)$
   d) $(6 - 2i) \div (4 + 5i)$
   e) $(3 + 5i) \div (4 - i)$

2. Find the reciprocal of each, and write it in the form $a + bi$.
   a) $3i$
   b) $2 - 7i$
   c) $-5 + 3i$
   d) $a + bi$

3. Solve for each x and y.

   a) $3x + yi = 7 + 3i$
   b) $2x + y + (4x - 3y)i = 3 + 11i$
   c) $(6 - 5i)x = (2 + 5i)$

Adaptations

The teacher may include the multiplication of three or more complex numbers, such as $(2 + 5i)(3 - 4i)(5 + 2i)$.

Other types of extensions could be similar to $\{ 4(3 - 2i) + 3(2 + 6i) \} (5 - i)$.

Determining the square root of a complex number can also be used as an activity to challenge some of the better students.

Example: Determine the square root of $(-7 + 24i)$.

Solution. Let $\sqrt{-7 + 24i} = a + bi$, where a, b are reals.

Square. $-7 + 24i = a^2 + 2abi - b^2$

Real = Real; Imaginary = Imaginary; therefore,
$-7 = a^2 - b^2$ $\quad$ $24i = 2abi$
Simplify to obtain
$12 = ab$, or $12/a = b$
Substitute, and solve.
$-7 = a^2 - (12/a)^2$
$-7 = a^2 - (144/a^2)$
$-7a^2 = a^4 - 144$
$0 = a^4 + 7a^2 - 144$
$0 = (a^2 + 16)(a^2 - 9)$, ... $a = 3, -3$
and correspondingly, $b = 4, -4$

The square roots $(a+bi)$ are therefore $(3+4i)$ and $(-3-4i)$. Check by squaring each root.

Concept E: Quadratic Equations

Foundational Objectives

- To demonstrate skill in solving quadratic equations (10 05 01). Supported by learning objectives 1, 2, 3, 7, and 8.
- To write a quadratic equation through analysis of the given roots (10 05 02). Supported by learning objectives 4, 5, and 6.

Objectives	Instructional Notes
E.1 To solve quadratic equations using the quadratic formula.	This topic was previously introduced in Objective D.8 of Mathematics A 30. It may be necessary to review briefly the methods of solving quadratic equations and re-introduce the quadratic formula. The teacher may have students develop the quadratic formula from the completing the square method of solving quadratic equations of the type $ax^2 + bx + c = 0$, or develop it for the students as a class activity. Students should obtain some practice in the use of the quadratic formula to solve equations. For any equation to be solved, the teacher could also expect students to draw the graph of the equation (replacing 0 by y), on a graphic calculator or computer. The students should be instructed to observe the roots of the equation and the x-intercepts of the graph. (TL)
E.2 To solve quadratic equations having complex roots.	Students could be given a quadratic equation whose roots are complex and be instructed to solve the equation. Write any complex roots in the form a+bi. The students should have little difficulty with this topic, as the previous unit dealt with complex numbers. Note that in many resource texts, these complex roots are referred to as non-real roots. Students should be made aware of these various terms being used to represent the same roots. (COM)

Examples/Activities

1. Solve each of the equations below, using the quadratic formula.

   a) $3x^2 - 5x - 2 = 0$
   b) $9x^2 + 6x + 1 = 0$
   c) $6x^2 + 10x + 3 = 0$
   d) $2x^2 - 8x - 3 = 0$
   e) $2x^2 - 7x + 2 = 0$
   f) $2x^2 - 17x + 21 = 0$

2. A rectangular picture has a length which is 1 more than twice its width. If the area of the picture is 136 cm^2, what are its dimensions?

3. A rectangular swimming pool is to be built having a depth of x m. The width is to be 1 more than twice the depth and the length is to be 4 more than three times the depth. If the surface area is 69 m^2, what is the depth of the pool? What are its other dimensions?

1. Determine the roots of each of the following equations. If the roots are complex, write them in the form a+bi.

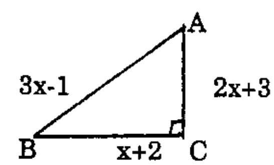
   a) $5x^2 + 2x + 4 = 0$
   b) $-3x^2 + 5x - 4 = 0$
   c) $4x^2 - \sqrt{3}\,x + 2 = 0$
   d) $7x^2 + 6x = -5$
   e) $8x = 6x^2 + 3$

2. Draw the graph for the function
   $4x^2 + 2x + 1 = y$. What are the x-intercepts?
   What kind of roots does this equation have?
   Draw the graphs of the equations in 1.a) and 1.b) above.
   What is the relationship between the roots and the x-intercepts? Would this be true for all such cases?

Adaptations

Students should also be given some exercises that allow them to utilize their knowledge of other areas of mathematics. An example might be the use of the Pythagorean Theorem.

Example.
Determine the value of x and the length of each side of the triangle below.



Objectives

E.3
To solve word problems involving real-world applications of quadratic equations.

Instructional Notes

Students can be given one or two real-world problems to work on as an introduction to this section. They should be allowed to work cooperatively, to discuss possible methods of solution, to restate the information given and asked for, and to develop the equation to be used in their solution. The solution obtained should be checked for reasonableness and written in statement form if it is the answer to a word problem.

The teacher should monitor this introductory activity, in order to assist students who need further direction.

Once the initial problems have been correctly solved, the teacher may wish to assign a few more of these types of problems for student practice.

Most resource texts have an adequate number of these types of exercises, but the teacher should attempt to assign exercises which provide some variety.

Examples/Activities

Solve.

1. The square of the third of three consecutive integers is equal to twenty-five more than the product of the first two. What are the three integers?

2. Two ships leave a port, A, at the same time, at right angles to each other. The first ship sails at a rate which is 1 knot faster than the second. After five hours, the ships are 145 nautical miles from each other. What is the rate of speed, in knots, of each of the ships?

3. The winning pair of drivers in an automobile rally from Victoria to Winnipeg (2 400 km) took 24 hours, 30 minutes to complete the trip. The first driver drove 900 km through the mountains, while the second driver drove the remaining 1 500 km at a rate of speed which was 50 kmh less than twice the rate of the first driver. What was the average rate of speed attained by each driver?

Supplementary 'wide-awake' question: What was the approximate average speed maintained for the entire trip?

Adaptations

The teacher should identify some problems from other areas that are applicable to this topic. They could then be introduced as a supplementary activity, or as examples to the students of some areas these situations are encountered. Again, many of these types of problems are available from resource texts.

Example 1.
If $n! = 72(n-2)!$, find the value of n, where $n \geq 0$.

Example 2.

From a rectangular sheet of metal 50 cm long by 36 cm wide, an open rectangular box is constructed by cutting squares of equal area from each of the four corners and folding up the ends. If the area of the base of the box so constructed is 1 320 cm^2, find the dimensions of the cut out squares. What is the volume of the box?

Other examples might include exercises from commerce, business, industry, or agriculture.

Concept E: Quadratic Equations

Objectives	Instructional Notes
E.4 To determine the nature of the roots of a quadratic equation using the discriminant.	The introduction to this section might be taken from the exercises completed in the previous three sections, or the teacher might provide or assign a set of equations which would result in roots which are of several different types. In either case, the teacher could instruct the students to categorize the roots as real or complex, to note how many different roots there are, and, if real, whether rational or irrational. Students could then be instructed to categorize their work in a chart, Equation | Type of Roots | Substitution in quadratic formula and then to analyze their summary to attempt to determine how to predict the nature of the roots expected for any quadratic equation. Allow a few minutes for discussion, before summarizing. If students are not able to identify the quantifier, $b^2 - 4ac$, known as the discriminant, the teacher should point out the value of the discriminant in each case. After introducing the terminology and the use of the discriminant in determining the nature of the roots of a quadratic equation, some exercises should be given for the students to practise.
	Students should be expected to demonstrate their understanding of the nature of the roots when the discriminant is equal to 0, is > 0, or is < 0.
E.5 To determine that the sum of the roots of a quadratic equation $ax^2 + bx + c = 0$ equals $(-b/a)$ and the product of the roots equals (c/a).	The teacher might introduce this section by having students solve a set of quadratic equations, writing the equation and the roots in a table similar to Equation | Roots and have students compare the roots and the coefficients of the equation. Note that the first few equations could have integral roots, to make the determination easier. When the students have understood the pattern, the teacher may formalize the results, namely that the sum of the roots of the equation $ax^2 + bx + c = 0$ equals $(-b/a)$ and the product of the roots equals (c/a). Students could be assigned a few exercises in which they are to determine the sum and product of the roots of a quadratic equation.

Examples/Activities

1. Determine the value of the discriminant in each of the following. What is the nature of the roots in each case?

   a) $4x^2 - 7x + 3 = 0$
   b) $2x^2 + 3x + 7 = 0$
   c) $-5x^2 + 10x - 5 = 0$
   d) $6x^2 + 9x - 2 = 0$
   e) $\sqrt{2}\,x^2 - 4x + \sqrt{8} = 0$
   f) $3ix^2 - 6x + 4i = 0$

2. For what value(s) of b would the equation $4x^2 - bx + 3 = 0$ have two real roots? 1 double root? (What would be the nature of this double root?) 2 complex roots?

3. For what value(s) of a would the equation $ax^2 + 6x - 2 = 0$ have two real roots? a double root? (what nature?) two complex roots?

Adaptations

This topic might be extended by supplying students with exercises and examples similar to the two below.

Example 1.

Describe the roots of $ax^2 + bx + c = 0$ if:
i) a is positive and c is negative.
ii) a and c are positive, $b = (1/2)a$.

Example 2.

A projectile fired upwards has a height given by $-4x^2 + 27x$ metres. Is it possible for this projectile to reach a height of 125 m?

---

1. Find the sum and product of the roots for each of the following equations. Determine the nature of the roots of each equation. Find the roots.

   a) $x^2 + 5x + 4 = 0$
   b) $x^2 - 7x = 44$
   c) $3x^2 - 10x + 3 = 0$
   d) $4x^2 + 7x - 2 = 0$
   e) $7x^2 + 8x + 2 = 0$
   f) $5x^2 - 3x + 4 = 0$
   g) $9x^2 - 12x + 4 = 0$

2. By inspection, determine the sum and product of the roots of the equation
   $4x^2 + 11x - 3 = 0$.

3. Determine the roots of the equation
   $2ix^2 - \sqrt{-3}\,x + 5i = 0$.

Have students verify the sum and product of roots by:

a) Adding

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

b) Multiplying

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Objectives	Instructional Notes
E.6 To write a quadratic equation, given the roots.	Given integral roots of an equation, the students should be instructed to write the quadratic equation which has the given roots. Students could be referred to their work in the previous section, if they are not able to write this equation in a short time. Once the equation has been written, and validated, other like exercises could be done, moving progressively to rational roots, irrational roots, complex roots, and double roots. Students should be instructed to formalize their procedure, noting that for $ax^2 + bx + c = 0$, $(-b/a) = r_1 + r_2$ and $(c/a) = r_1r_2$, where r_1 and r_2 are the roots of the equation. Therefore, the equation is written in initial form as $x^2 - (r_1 + r_2)x + r_1r_2 = 0$ and then simplified by clearing any denominators. Practice exercises could then be assigned by the teacher.
E.7 To solve equations of degree greater than two by expressing them in quadratic form. E.g.: $x^4 - 34x^2 + 225 = 0$.	The students can be given some exercises of this type to complete in pairs, or individually. They might need some instruction or hints as to what is expected of them in solving particular cases. Some examples may have to be readied in advance, in order to illustrate some of the techniques. None of the techniques employed here are new to the students but their use in this context may not be expected.

Examples/Activities

1. Write the quadratic equation which has the given roots.

   a) {6,-1}
   b) {4, 7}
   c) {2/3, 1/2}
   d) {3/5, -1/6}
   e) {$\sqrt{3}$, $-4\sqrt{3}$}
   f) { $2 \pm \sqrt{5}$}
   g) { 6 - 3i, 4 + 5i}
   h) {-5/3}, double root.
   i) { h/3, k/2}
   j) { -3π, -5π}

Adaptations

An extension to this topic is to provide other types of questions that deal with the sum and roots of a quadratic equation, similar to the example below.

Example.

Determine the values of m and n in the equation $4x^2 + mx - 2n = 0$, if the sum of the roots is known to be -3/4 and the product of the roots is -5/2.

1. Solve each of the following equations. Be sure to check your answers to identify the solutions in each case.

   a) $x^4 - 13x^2 + 36 = 0$
   b) $x - 3\sqrt{x} - 4 = 0$
   c) $4x^4 - 37x^2 + 9 = 0$
   d) $8x^{-2} + 6x^{-1} + 1 = 0$
   e) $36x^5 - 25x^3 + 4x = 0$
   f) $(x^2 + 3)^2 - 5(x^2 + 3) = 6$

Students could be asked to graph each of the equations in the left-hand column and to identify the x-intercepts in each case. The graphing could be done using graphic calculators or computers and the intercepts located through the use of the trace and zoom functions. (TL)

Another extension of this topic could be through the use of a few word problems similar to the example given below.

Example.

A particle travels a distance from the x-axis best described by the equation $4x^5 - 677x^3 + 169x = 0$. For what values of x will this particle be above the x-axis?

Objectives	Instructional Notes

E.8
To solve quadratic inequalities.

The teacher may wish to provide students with a sample quadratic inequality such as $x^2 - 4x - 21 \geq 0$ and instruct the students to find all possible values of x that would make this a true statement. The teacher would then ask the students for their answers, with justification of their solution. A discussion about the various procedures may provide the students with the insight needed to solve such inequalities. (COM)

Alternatively, the teacher may wish to model several different methods that could be used to solve quadratic inequalities. Some of these are stated below.

Method 1. Trial and error. The student substitutes a number and determines whether it works. After many trials, the student is able to reach an almost exact conclusion.

Method 2. Analysis of signs and factoring.
Example. $x^2 - 4x - 21 \geq 0$
Factors are $(x - 7)$ and $(x + 3)$. Result is positive; therefore, both factors must be positive, or both must be negative.
If $x - 7 \geq 0$, and $x + 3 \geq 0$, or $x - 7 \leq 0$ and $x + 3 \leq 0$,
$x \geq 7$ and $x \geq -3$, or $x \leq 7$ and $x \leq -3$,
So $x \geq 7$ or $x \leq -3$

Method 3. Factor the inequality and set each factor equal to zero. Plot these critical points on a number line. These points divide the number line into sections. Take a test point from each section and determine if it makes the statement true. After a few examples, students find they need only one test point. Why? A completed example is in the next column.

A chart is useful for Methods 2 and 3.

Examples/Activities

1. Find the values of the variable which will make each inequality a true statement. Use any method.

   a) $x^2 + 5x - 14 < 0$

   b) $x^2 - 6x - 16 > 0$

   c) $2x^2 - 7x + 3 \geq 0$

   d) $3x^2 + 10x + 3 \leq 0$

   e) $6x^2 - x - 15 \geq 0$

   f) $4x^3 - 8x^2 - 12x \leq 0$

   g) $9x^2 + 6x + 1 > 0$

2. Solve $4x^2 + 9x - 9 < 0$, using method 3.

   Solution. Factors are (4x-3) and (x+3).
   Find critical points by setting the factors equal to zero.
   Critical points are x = 4/3 and x = -3. Plot these on a number line. Use open circles as they are not included in the solution.

   -3 4/3

   Take a test point less than -3 (e.g.: -5). Does it make the statement true? No. Take a test point between -3 and 4/3 (e.g.: 0). Does it work? Yes. Shade this section of the number line. A test point greater than 4/3 does not work.
   The number line now looks like

   -3 4/3

   The solution is $-3<x<4/3$.

Adaptations

This topic may be extended by introducing more examples of quadratic inequalities of degree greater than two. As well, there may be some word problems in your main text resource that you might use in your class.

Concept F: Polynomial and Rational Functions

Foundational Objectives

- To demonstrate the ability to graph and to analyze the graphs of polynomial and rational functions (10 06 01). Supported by learning objectives 1 to 3.
- To demonstrate understanding of an inverse of a function (10 06 02). Supported by learning objectives 4 and 5.

Objectives	Instructional Notes
F.1 To define and illustrate polynomial and rational functions.	In order to work effectively on this unit, students should review some topics they have studied previously. The Factor Theorem, the Remainder Theorem, and synthetic division can all be utilized in this unit. A brief review of some or all of these topics may be a good starting point. In addition to these, the students must become familiar with the definitions associated with polynomial and rational functions. A polynomial function is usually described as $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots\dots\dots + a_1x + a_0$. Polynomial functions are generally written in descending order, and a_n is referred to as the leading coefficient. The rational functions are usually described as $f(x) = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$ and both p(x) and q(x) are polynomials. It is recommended that a selection of resource texts be used in this unit, in order to provide adequate coverage of the concepts presented.
F.2 To sketch the graphs of polynomial and rational functions with integral coefficients, using calculators or computers.	In this section, it is expected that students will be given the opportunity to graph a series of polynomial and rational functions, utilizing graphic calculators or computers. It will be necessary for the students to demonstrate their capability in using these technologies. Because of this, the teacher may wish to have students complete a series of graphs beginning with linear functions, in order to build student confidence in using this technology. If a printer is attached, have some of these graphs printed for documentation and for further work in the next sections. If no printer is available, have students make a sketch of the graph in their notebooks, asking them to note the shape and any intercepts on the x- and y- axes. (TL)

Examples/Activities

1. Factor each of the following polynomials.

   a) $x^3 + 2x^2 - x - 2$
   b) $x^3 - 3x^2 - 10x + 24$
   c) $x^4 + 2x^3 - 7x^2 - 8x + 12$

2. Determine the remainder in each case.

   a) $(x^3 - 4x^2 + 7x + 2) \div (x - 3)$
   b) $(4x^4 + 5x^2 - 3x - 7) \div (x + 4)$

3. Determine three ordered pairs for each of the polynomial functions given below. You may wish to use the remainder theorem.

   a) $f(x) = 4x^3 - 3x^2 + 5x - 2$
   b) $f(x) = 6x^4 - 4x^3 + 5x^2 - 4x + 3$

4. Determine three ordered pairs for the given rational function.
   $f(x) = \dfrac{3x - 5}{4x^2 - 9}$

5. a) Create a polynomial function of at least four terms. Determine three ordered pairs for your function.
   b) Create a rational function. Determine three ordered pairs for this function.

1. Graph each of the following functions on a graphic calculator or computer. Sketch the result in your notebook noting any intercepts.
   a) $f(x) = 2x - 5$
   b) $f(x) = -1/3x + 5$
   c) $f(x) = x^2 - 4x + 3$
   d) $f(x) = -2x^2 + 5x - 2$
   e) $f(x) = -x^3 + 4$
   f) $f(x) = x^3 - 5x + 4$
   g) $f(x) = x^3 - 3x^2 - 10x + 24$
   h) $f(x) = -x^4 + 3x^2 - 2$
   i) $f(x) = x^4 + x^3 - 3x^2 - x + 2$
   j) $f(x) = \dfrac{-1}{4x - 3}$
   k) $f(x) = \dfrac{2x + 5}{3x^2 - x - 2}$

Adaptations

At this stage of the unit, all students should be expected to demonstrate they have learned the basic definitions associated with these types of functions. They might demonstrate this through stating the definitions, or by providing examples that demonstrate the concept.

Students may also be expected to determine factors, remainders, and ordered pairs by employing available technology, as well as by calculation. (TL)

After graphing functions of these types on calculators and computers, some students might be asked to plot one of these types of graphs by determining ordered pairs. These students would have to work on their own to determine how many points are needed to define the graph of the function and to graph it. (TL)

Students might also be asked to search through available math-related resources to note any polynomial functions that are referred to in real-world situations, and in what context. Some examples may be found in commerce, or engineering texts. (TL)

Objectives

F.3
To analyze the characteristics of the graphs of polynomial and rational functions and to identify the 'zeros' of these graphs.

Instructional Notes

The resource texts that include polynomial functions also include the major characteristics that define them and information on how they may be analyzed.

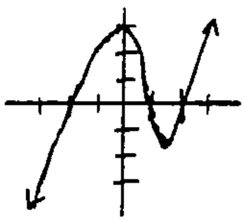
The students should refer to the graphs already drawn in previous sections and be instructed to note: i) the degree of each function, and the number of points of intersection with the x-axis, ii) the leading coefficient, and the direction of the graph from left to right, iii) the number of times the graph changes direction, and the signs in the function, iv) the gaps in the graphs where asymptotes occur, and restricted domains, and v) any y- intercept, and the constant in the function.

Given a set of graphs representing polynomial functions, they should be able to identify the major characteristics of the graph. As well, given a set of polynomial functions, they should be able to identify the characteristics of the function and to provide a sketch of the function. The process of actually sketching a function is detailed in the resource texts that include polynomial functions.

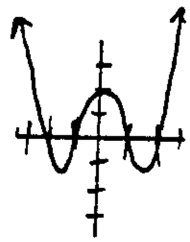
Examples/Activities

1. List the major characteristics (see previous column) of each of the graphs representing polynomial and rational functions as drawn below.

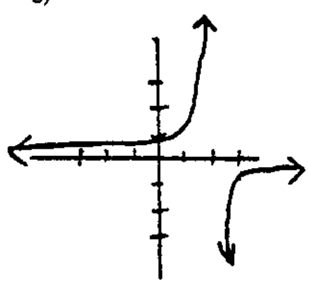
a)



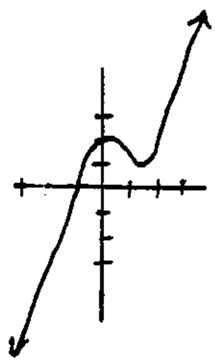
b)



c)



d)



2. Given the following functions, identify the major characteristics, and then sketch the graph of the function, using these as a guide.

   a) $f(x) = x^3 - 3x - 2$

   b) $f(x) = -x^3 + x^2 + 7x - 7$

   c) $f(x) = \dfrac{1}{x^2 - 4}$

Adaptations

Students might be expected to use the Remainder and Factor Theorems, and synthetic division, to aid in finding the roots (and the number of roots to expect) of the given functions, or in developing a table of values for the functions. They could also be introduced to Descartes' rule of signs. These could be combined to expect a more accurate sketch of the graphs of polynomial functions.

The use of terms such as concave up and concave down can be utilized to make descriptions more illustrative.

Objectives

F.4
To define, determine, and sketch the inverse of a function, where it exists.

Instructional Notes

Students have studied inverses prior to this topic, but not the inverse of a function. They should be made aware of the definition of the inverse of a function and the symbol associated with it. If f(x) is a function, then its inverse is symbolized by $f^{-1}(x)$. It is important that the students understand that this is a symbol to denote the inverse of the function and not a reciprocal.

The starting point for this topic is usually to employ the definition of the inverse of a function. The definition varies somewhat depending on the resource used, but the following is generally acceptable.

f(x) and $f^{-1}(x)$ are inverses if for every $(x,y) \in f(x)$, then $(y,x) \in f^{-1}(x)$.

Alternatively, you might use $f^{-1}(f(x)) = f(f^{-1}(x)) = x$

The determination of the inverse of a function can be done using this definition. However, it may also be accomplished by interchanging the variables in the function and solving for y, which is the method utilized in most of the text resources.

In order to sketch the graph of the inverse of a function, first graph the function and then its image as reflected in the line given by y = x. This reflection also allows you to determine quickly whether two graphs represent inverses of each other.

Note that not every function has an inverse which is itself a function. This only occurs when f(x) is one to one. In general, $f^{-1}(x)$ is a relation.

Examples/Activities

1. For each f(x) given below, determine $f^{-1}(x)$.

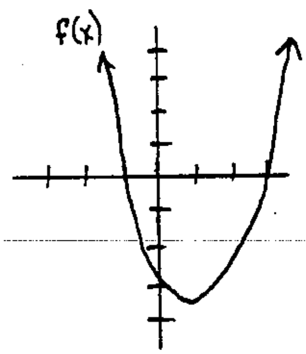
   a) $f(x) = 2x + 5$
   b) $f(x) = -4x + 3$
   c) $f(x) = 3x^2 - x$
   d) $f(x) = \{(-2,5), (4,-2), (1,5), (0,-1)\}$
   e) $f(x) = 3x-2$, where $x \in \{1, 2, 3, 4, 5\}$
   f) $f(x) = 4x^2 - 3x + 4$

2. Determine the inverse of each of the following functions. Sketch each function and its inverse.

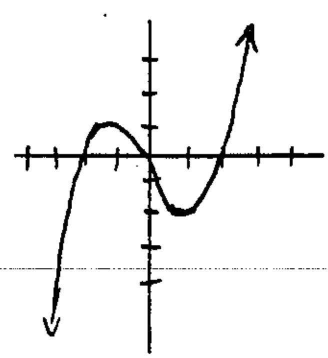
   a) $f(x) = 4x - 1$
   b) $f(x) = -3x^2 + 2x - 4$
   c) $f(x) = 4x^3 + 4x^2 - x - 1$

3. Given each of the following graphs representing f(x), sketch the graph of $f^{-1}(x)$.

   a)



   b)



Adaptations

This topic can be extended by having students find the inverses and sketch the graphs of other types of functions. Some examples are given below.

Example 1.

Find the inverse of $y = |x|$, and sketch y and y^{-1}.

Try a similar exercise for $y = 2|x-3| - 4$.

Example 2.

Find the inverse of $y = \sqrt{3x-1}$, and sketch both y and y^{-1}.

Other types of exercises could include similar exercises involving composite functions.

Objectives

F.5
To define, determine, and sketch the reciprocal of a function.

Instructional Notes

Students should learn the basic definition of a reciprocal function, that is, if f(x) represents a function, then $\frac{1}{f(x)}$ is its reciprocal function. They should also be made aware that the roots of f(x) represent critical points for the reciprocal function, in that these points indicate where vertical asymptotes will occur. They should be given practice in determining these critical points, perhaps before attempting any other graphing exercises in this section. Students should also have some practice in determining any horizontal asymptotes that may exist.

Students could be introduced to this topic using a table of values for a linear equation, such as y = 2x -1. They could be instructed to complete a table of values having at least three ordered pairs. Then they could be instructed to determine the reciprocal function, complete a table of values, and sketch. The graph of the reciprocal function could also be done using available technology. (TL)

For y = 2x - 1

x	-2	-1	0	1	2
y	-5	-3	-1	1	3
1/y	-1/5	-1/3	1	1	1/3

Examples/Activities

1. Determine the reciprocal function for each of the following.

   a) $f(x) = 4x + 3$

   b) $f(x) = 3x^2 - 7$

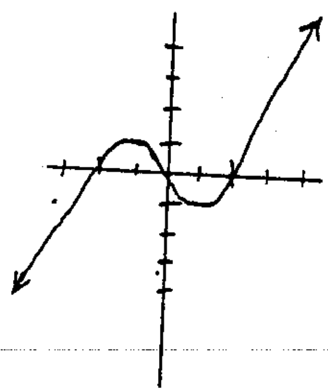
   c) $f(x) = x^3 + x^2 - 4x - 4$

   d) $f(x) = \{(-3,4),(-2,3),(-1,1),(0,-3),(1,-9)\}$

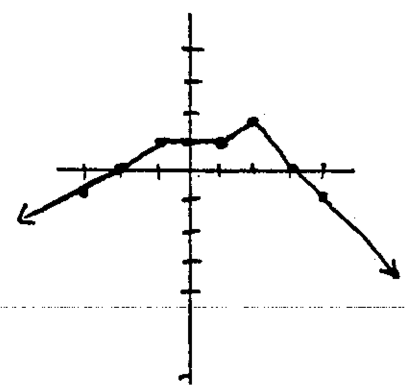
2. Sketch the graph of each function in question 1. Then sketch the graph of the reciprocal functions as determined in question 1.

3. Sketch the reciprocal function of each of the following.

   a)



   b)



Adaptations

The topic of reciprocal functions could be related to the six trigonometric functions, as studied in Mathematics A 30 (Objective G.6), and how these reciprocal functions could be utilized to simplify calculations in some specific questions.

Concept G: Exponential and Logarithmic Functions

Foundational Objectives

- To develop skills and knowledge in working with a variety of exponential and logarithmic functions (10 07 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to apply the knowledge of exponential and logarithmic functions to real-world situations (10 07 02). Supported by learning objectives 6 to 17.

Objectives

G.1
To define exponential functions and logarithmic functions.

Instructional Notes

Students could be presented with some real-world examples, such as chain letters, the checkerboard problem, or interest calculations (see next column), to show how exponents are encountered in various situations. These all represent exponential functions. From these examples, the definition of an exponential function could be introduced. A general form is $f(x) = a^x$, where $a > 0$, and $x \in R$.

The logarithmic function can be introduced as the inverse of the exponential function. If $y = a^x$ is the exponential function, then the logarithmic function can be written as $x = a^y$. This can then be written in logarithmic form as $y = \log_a x$. E.g.: $\log_a x$ is the power to which we raise a to get x. Students can be given some practice converting from one form to the other.

Examples.

$8 = 2^3$ is equivalent to $\log_2 8 = 3$

$c = b^a$ is equivalent to $\log_b c = a$

$2^{\log_2 3} = 3$

$5^{\log_5 q} = q$

Examples/Activities

1. A chain letter is sent out asking that the recipient copy the letter and send it to four other people. Assuming no one breaks the chain, set up a table of values that shows the number of letters in the chain at each stage, for six stages. Write the exponential function that represents this situation.

2. According to an old story, a king is to reward a loyal subject by placing a penny on the first square of a checkerboard, two pennies on the second square, four pennies on the third square, and continue this pattern until the 64th square. Write an expression that indicates how many pennies there will be on any square. How many will there be on the last square?

3. If interest is added only once a year, write an expression for the amount of interest earned on an investment of $1 000 at a rate of 7% compounded for any number of years. How much would there be in ten years?

4. Rewrite each of the following exponential functions in logarithmic form.

   a) $y = 3^x$

   b) $y = (1/2)^x$

   c) $y = 10^x$

5. Rewrite as an exponential function

   a) $y = \log_7 x$

Adaptations

Students might be introduced to the particular exponential function $f(n) = (1 + 1/n)^n$ and to the limit of the values of this function as n approaches ∞.

This is the irrational number **e**, named after Euler, and is one of the possible values for the base a in the formula $y = a^x$.

The exponential function $y = f(x) = e^x$ has many applications in nature.

Several of the resource texts make use of this number **e** and have real-world problems based upon **e**. You might have students research the catenary curve (which approximates a cable suspended between two fixed points, $y = \frac{e^x + e^{-x}}{2}$, where it is used, and by whom.

Also useful is

$y = e^{-x^2}$

which has many applications in statistics.

Objectives	Instructional Notes
G.2 To use correctly the laws of exponents for integral and rational exponents.	This section includes the review of the laws of exponents for integral and rational values, the extension of these laws to include irrational exponents, the solution of equations involving exponents, and the introduction of the laws of logarithms for base ten. It is important that students gain an understanding of the laws of logarithms and how they relate to the laws of exponents. The solution of equations involving exponents, by converting to the same base, is also useful in further work, especially series. The teacher may wish to structure the review of laws of exponents as a series of exercises designed to allow students to review these on their own. The introduction of irrational exponents could be done by pairs of students working on a few exercises, and discussing their procedures with each other. (COM) The solution of equations could be done by example, allowing students to propose possible procedures, and then checking the result, while the laws of logarithms could be introduced by the teacher as a class lesson, using didactic questioning. The laws of logarithms should include those for: multiplication ($\log mn = \log m + \log n$), division ($\log x/y = \log x - \log y$), powers ($\log x^n = n \log x$), and roots ($\log \sqrt[n]{x} = 1/n \log x$).

Examples/Activities

1. Simplify each of the following.

   a) $x^3 \cdot x^4$ b) $(x^5)^3$ c) $x^{-5} \cdot x^2$
   d) $x^7 \div x^4$ e) $x^{\sqrt{3}} \cdot x^{2\sqrt{3}}$ f) $(x^{\sqrt{6}})^{\sqrt{3}}$
   g) $3x^{2\pi} \cdot 4x^{3\pi+2} \cdot x^{-\pi}$

2. Solve each of the following equations for the variable.

   a) $(3)^{4x} = (27)^{2x-1}$ b) $(4)^{3x} = (8)^{x+5}$

   c) $(1/32)^{2x-3} = (64)^{-2x+7}$

3. Write as a single logarithm. Simplify.

   a) log 9 + log 6 b) log 42 - log 14

   c) 3 log 4 + 2 log 2 d) 4 log 7 - log 7

   e) log x + 2 log y - 1/3 log z

   f) 3 log (x-2) - log (x-2)

   g) log (x^3 - $3x^2$ -4x) -{log(x-4)+log(x+1)}

4. Many other varieties of exercises are available from resource texts. These include problems based on real-world situations.

Adaptations

Students might be shown how common logarithms were used for calculation before the advent of calculators and computers. This should be a demonstration simply for historical purposes and should not be considered part of the course, nor should it be tested. The slide rule could also be shown, as an example of how mathematicians worked before calculators became feasible. It is not necessary to demonstrate how to use the slide rule, but simply to indicate it is based on logarithms and that calculations were done in that manner. (TL)

Concept G: Exponential and Logarithmic Functions

Objectives	Instructional Notes
G.3 To work with logs of numbers with bases other than 10.	In this section, the laws of logarithms should be adapted to allow for work with bases other than 10. The exercises and examples chosen should be similar to those in the previous section. The teacher may wish to begin by having the students discuss the solutions to log statements such as $\log_2 8 = ?$, $\log_6 (1/216) = ?$, $\log_a a^b = ?$ The introduction of $\log_a n = \frac{\log_b n}{\log_b a}$, as a means of changing bases, might also be introduced at this point.
G.4 To construct graphs of exponential functions and logarithmic functions, to identify the properties of these graphs, and to recognize they are inverses of each other.	Students should be given a series of graphs to sketch on a graphic calculator or computer, transcribing the general shape to paper, or obtaining print-outs for further analysis. The first such series of graphs could be $y = a^x$, where $a>0$. The second series could be $y = \log_a x$. These two families of graphs could be compared to determine some characteristics that each type has in common, and the relationship between the two types. Other types of exponential and logarithmic functions could then be graphed, using graphic calculators or computers. The teacher may wish students to obtain a table of values for one or two of the first graphs, in order to understand the calculations that are necessary before the graphs appear on a screen.

Examples/Activities

1. Write as a single log statement.

   a) $\log_3 7 + \log_3 6$
   b) $\log_5 (x-3) + \log_5 4$
   c) $6 \log_7 2$

2. Solve for the variable.

   a) $\log_7 (3x+2) + \log_7 2 = \log_7 16$
   b) $\log_6 (x^2-9) - \log_6 (x+3) = \log_6 5$
   c) $\log_3 (x-6) + \log_3 (x) = 3$

3. Change $\log_6 5$ to a logarithm written in base 10.

4. Convert $\log_3 8$ to a base 6 log.

There are many other types of exercises and real-world problem solving situations in many of the resource texts. Teachers should also utilize these.

1. Graph the following, using a table of values obtained by calculator or computer.

   a) $y = 3^x$
   b) $y = \log_3 x$

2. Graph the following, using a calculator or computer. Record results for further analysis.

   a) $y = 2^x$ b) $y = 5^x$ c) $y = 8^x$
   d) $y = (1/2)^x$ e) $y = (3/4)^x$ f) $y = (1/8)^x$
   note d) could also be $y = 2^{-x}$
   g) What happens when $y = 1^x$?

3. Graph each of the following. Record the results for further analysis.

   a) $y = \log_5 x$ b) $y = \log_{12} x$ c) $y = \log_2 x$
   d) $y = \log_{(3/4)} x$ e) $y = \log_{(1/2)} x$

4. a) What happens to graphs of exponential functions when the exponent x > 1 ? = 1? between 0 and 1?
   b) From this, what might happen to a sum of money that is invested at 7% over a long term (assume no inflation)?
   c) Some Las Vegas casinos point out that certain games pay back as much as 98% of money gambled. From the graphs, what might you conclude would happen? (CCT)

5. Repeat 3.a) for logarithms.

Adaptations

Teachers could introduce the inverse of the exponential function $y = e^x$, $y = \log_e x$. Note that $\log_e x$ is commonly referred to as ln x, the natural logarithm of x. Most student calculators will have this function. Students can be given some exercises in which they are able to use the ln button on their calculator. This function also has many real-world applications. Some of the questions suggested in the resource texts should be attempted by the students.

Given calculators or computers, students could be given a series of functions to graph that represent other exponential and logarithmic functions than the straight-forward ones in the previous column.

For example, students could plot the rate of growth of a thirty year bond fund which has a rate of annual interest of 6%? (NUM)

Other types of graphs students should do are those representing the families suggested by $y = e^x$, and $y = \ln x$. They might also graph functions such as

$y = e^{-x^2}$
and $y = (e^x + e^{-x}) \div 2$.

Many other graphs representing exponential and logarithmic functions taken from real-world examples could also be graphed. Most text resources have an adequate supply of these types of questions.

Discuss properties, domain, range, x and y intercepts, increasing and decreasing intervals, for exponential and logarithmic functions. What point always belongs, what is the role of the base?

Objectives	Instructional Notes
G.5 To sketch graphs of exponential and logarithmic functions by selecting an appropriate point for the new origin.	This section is intended to have students become familiar with exponential and logarithmic graphs that have the same properties as those in the previous section, but have constants included which cause shifts in the graphs. By choosing a different point as the origin, the shifts can be counteracted and the same properties become much more evident. Once students understand how to shift or translate the axes, a few exercises might be given for them to graph. These exercises might be relatively simple, such as graphing $y = 3 + 2^x$, and then shifting the axes, or they might be more difficult, involving several shifts, such as $y = 2 + 2^{x-3}$. This could also be written in the form $y-2 = 2^{x-3}$. Similar types of graphs could be done for logarithmic functions.
G.6 To solve exponential and logarithmic equations.	This section could be introduced by asking students to attempt to determine how long it would take an investment to quadruple in value if it grew at an annual rate of 8%. Students could work individually, in pairs or in small groups to discuss strategies, pertinent information, how to make use of exponents or logarithms, and attempt to solve the problem. Should the students seem to be having difficulty starting, a second problem might be posed. The second problem could be stated as; "What number, when cubed, has a result of 500?" Once students are able to solve this, they may return to the first problem. (CCT) Alternatively, the teacher may wish to do a class example of an exponential or logarithmic equation and then have students complete a few exercises.

Examples/Activities	Adaptations
1. Graph each of the following. a) $y = 4 + 3^x$ b) $y = -3 + 2^x$ c) $y = 1 + 4^{-x}$ d) $y = -3 + 2^{x+2}$ e) $y = 5^{-x+1} - 3$ Identify the type and magnitude of the shift in each case. Draw the graph with a new origin that takes the shift into account. 2. Draw the graph in each case. Identify the nature and magnitude of the shift. Redraw the graph using a new origin that takes the shift into account. a) $y = 4 + \log_3 x$ b) $y = -2 + \log_5 x$ c) $y = 1 + \log_7 (x-4)$ d) $y = -3 + \log_4 (3 - x)$	As in the previous sections, the introduction of similar questions involving $y = e^x$ and $y = \ln x$ could be used to adapt or extend this section. Exercises which include shifts could be given to the students for similar type of practice in translating the axes to take into account these shifts.
1. Find the value of x in each of the following. a) $2^x = 6$ b) $3^x = 18$ c) $4^x = 6^{x-1}$ d) $2^{-x} = .048$ e) If $7 500 was invested at 7% compounded annually and the present amount including interest is $12 450, how long has this principal been invested? 2. Solve for the variable in each of the following. a) $\log_7 (x-3) + \log_7 (x+5) = 2$ b) $\log \sqrt[4]{x} = \sqrt{\log x}$ c) $pH = -\log H^+$, where H^+ is the concentration of the hydrogen ion in moles per litre. A batch of tomatoes has $H^+ = 6.2 \times 10^{-5}$. What is its pH value? A piece of blackboard chalk has $H^+ = 1.5 \times 10^{-8}$. What is its pH?	This topic could be extended by having students find the natural logarithms of given numbers. Example 1. Find ln 7.54 Example 2. Find ln 68.8 Example 3. Find x given $x = e^{0.456}$ Other types of questions that could be dealt with in this section include the common business 'fact' of the rule of 72, that the doubling rate of an investment is the point at which the rate of interest multiplied by the time in years is 72. Students could determine the accuracy of this rule of thumb by using calculators. (NUM)

Concept G: Exponential and Logarithmic Functions

Objectives	Instructional Notes
G.7 To solve word problems involving exponential and logarithmic functions.	Most of the text resources have adequate numbers of real-world problems. The teacher may wish to have the students work on a few of these in pairs and have the entire class take part in the 'taking-up' of these initial exercises. Most of these problems deal with growth and decay, and require formulas in some cases. These formulas are normally part of, or immediately preceding, the problems given in the resource texts. Adaptation of these problems to events and places closer to the student might generate more interest, but will take extra preparation time. When students have had some successful experience with the initial exercises, they might be provided with some further problems for practice. (IL)
G.8 To identify a geometric sequence.	Begin with a brief review of the definition of sequence and some of the terms associated with an arithmetic sequence as taken in Mathematics 10. Once the review is complete, students could be given some examples of geometric sequences and asked to determine the next few terms of the sequence. When they have completed these, the definitions and terms associated with geometric sequences can be formalized. The basic terms are the first term, the common ratio between successive terms, and the number of terms. Various resources employ different variables to designate each, but the definitions and usage remain constant. Student exercises for this topic can be identification exercises, completing the next few terms, or the generation of geometric sequences.

Examples/Activities

1. A formula to describe population growth in Canada is $P = P_i e^{nt}$, where P_i is the initial population, P is the population at time t, and n is a constant representing the rate of growth. If the population of Canada in 1981 was 25 000 000, and in 1991, it was 27 500 000. Find the value of n and then use this value to predict the Canadian population in 2001.

2. Using the formula in number 1, and knowing that the population of Saskatoon was 120 000 in 1961, and 190 000 in 1991, find the value of n and then use this value to predict Saskatoon's population in 2006.

3. If an investment of $12 500 was made in 1975, having an annual rate of interest of 7%, how much is that investment now worth? At what point did this investment double?

1. Given the following sequences, identify those which are geometric sequences. For each geometric sequence, also identify a (t_1), and r, and write the next two terms of the sequence.

   a) 4, 7, 10, 13, ...
   b) 2, 3, 4.5, 6.75,........
   c) 1, 1, 2, 3, 5, 8, 13,........
   d) 2, -6, 18, -54, ...
   e) 2x, 4x, 6x, 8x, ...
   f) $\sqrt{3}$, $2\sqrt{3}$, $3\sqrt{3}$, $4\sqrt{3}$,..........
   g) 12, 6, 3, 1.5, ...
   h) x, $2x^2$, $4x^3$, $8x^4$,............
   i) -5, 10, -20, 40,...........
   j) $\sqrt{2}$, 4, $8\sqrt{2}$, 32, ...

2. Given the geometric series 1.2, 1.8, 2.7, 4.05, ..., identify r. Explain how you obtained the value of r.

3. For the geometric sequence at, at^4, at^7, at^{10}, find the common ratio. Justify your answer.

Adaptations

The teacher might ask students to identify some geometric sequences that represent real-world situations. The students could be asked to state these in a sentence, provide the first three or four terms, and identify a and r. Some examples might include cell division, number of ancestors, interest on savings bonds, the successive heights of a bouncing ball, chain letters, and pyramid organizations. (COM)

The teacher may also have students plot some geometric sequences where $|r| > 1$, join the plotted points with a smooth line, and observe the result. Repeat with geometric sequences where $0 < |r| < 1$.

Objectives	Instructional Notes
G.9 To determine the nth term of a geometric sequence.	The teacher may initially instruct students to determine a specific term of a given sequence, without further instruction. Students could work individually or in pairs on a short series of exercises of this nature. The exercises could be designed so that the students can begin to identify a procedure other than continually multiplying successive terms by r. If the class is not able to identify the general rule, the teacher can help them formalize the process by demonstrating the rule for finding the nth term of a geometric sequence through class examples. Students should be able to demonstrate their understanding of the general rule by correctly applying it to a short series of assigned exercises where they are instructed to find the nth term, or by being able to develop the rule from the general geometric sequence $a, ar, ar^2, ar^3, \ldots\ldots\ldots$
G.10 To calculate the required number of geometric means between given terms.	Students could work individually, in pairs, or small groups, and have a problem posed to them. The information provided would allow them to determine a, and t_n. They would be instructed to determine a specific number of terms (geometric means) between the two. They would be expected to provide the solution and justify their answer. When this exercise has been completed, the entire class can work together to formalize the procedure, including specific definitions needed for this topic. A short series of exercises on calculating means could then be given as an assignment.

Examples/Activities

1. Find the indicated term in each of the following.

   a) sixth term of 1, 3, 9, ...
   b) t_5, of 2, -4, 8,.......
   c) t_7, 12, 6, 3, ...
   d) t_6, where $a = 4$, $r = .5$
   e) t_7, where $a = \sqrt{5}$, $r = \sqrt{3}$
   f) t_6, where a = $2 500, r = 1.1*
   *What % interest rate does 1.1 represent ?

2. A student is researching her family genealogy and finds that she is able to record all direct ancestors to the ninth generation preceding her. How many ancestors would she have in this ninth preceding generation ?

3. It is known that the half-life of a certain isotope is three days. If a research lab receives 2 kg of this isotope, how many grams will remain three weeks later?

Adaptations

Students could be instructed to use calculators or computers to find the value of long-term investments.

Example. In 1985, a savings bond was purchased which offered an annual rate of interest of 9% for thirty years. If the initial purchase price of the bond was $5 000, how much should it be worth in 2015? (NUM)

Students can also be given other types of exercises in which geometric series with a, r, and ar^{n-1} are involved. In these, students are given two of the values, and asked to determine the third.

Example 1. Find the value of a, if $r = 1/3$, and $t_6 = 2/81$.

Example 2. Find the value of r, if $a = 3$,and $t_7 = 192$.

Example 3. Find the value of a if $t_6 = 128/243$ and $t_3 = 16/9$.

---

1. Given that the first term of a geometric sequence is 12, and the last term is 4/27. Find three geometric means for this sequence.

2. If $a = -32/125$, and $t_6 = 25$, find the four geometric means for this sequence.

3. Find three geometric means between 9 and 208 1/3.

4. Use a calculator for the following problem.

Florence invested $2 750 in a term deposit at an unknown rate of interest compounded annually. Six years later, the term deposit was worth $4 127. Find the rate of annual interest earned, and the value of the term deposit in each of the five intervening years.

When students are solving for r in determining the geometric means, have them utilize logarithmic equations as well as equations where they set both bases equal to each other. The use of logarithmic equations will reinforce the utility of logs and will also allow them to solve questions such as number 4 in the preceding column.

Most of the resource texts will have real-world problem situations that can be assigned for further student practice. (IL)

Objectives

Instructional Notes

G.11
To calculate the sum of a geometric series.

The definition of a geometric series and the formulas required for geometric series should be introduced. Some of the better students may wish to derive the formula on their own, given a starting point. However, for most students, the teacher could lead the class through the derivation.

The class should practise using these formulas to find the indicated sum of a few series. The summation notation for geometric series should also be introduced in this section. The students should become familiar with all terms associated with the summation notation.

The class should be given a variety of exercises that allows them to practise using these formulas and notations.

G.12
To define and illustrate the following terms: geometric sequence, compound interest, present value, annuity, geometric means.

This section is placed here in order for students to take inventory of some of the terms and concepts they will need in the following sections, where applications of their knowledge will be expected. Students should be able to define and explain each of the terms listed here, as well as demonstrate their knowledge of the terms.

In this section, the specific terms of annuity and present value are the only ones not encountered previously in mathematics, and should be introduced to the class. Most of the students will have encountered these terms through their everyday life, or in other classes, and may be able to provide background explanation, and know examples of applications. (COM)

Examples/Activities

1. Find the indicated sum in each case.
   a) $a = 6, r = 2, n = 7$
   b) $a = 14, r = .5, n = 6$
   c) $a = 36, r = -1/3, n = 8$
   d) $a = 4, r = 3, t_n = 48$

2. Find the indicated variable in each case.
   a) $r = 1/2, n = 7, S_n = 190.5, a = ?$
   b) $a = 1/3, r = 2, S_n = 21, n = ?$
   c) $S_n = 170.5, r = -.5, n = 10, a = ?$

3. Evaluate.
   a) $\sum_{i=1}^{9} 3(2)^{i-1}$
   b) $\sum_{j=1}^{5} 3(5)^{j-1}$

4. Write the following in summation notation.
   $3 + 6 + 12 + 24 + ... + 96.$

5. The sum of $-2 + 6 - 18 + ... + 486$ is ____.

Adaptations

As an alternative, the teacher might find some real-world problems that incorporate most of the concepts used in the summation of a geometric sequence.

One example that might be used is the story problem following.
Example.

A Saskatchewan fur trader of the 1700s left a will that stipulated his estate should remain untouched for two hundred years and then be shared equally by each of the remaining direct descendants. The estate at the time of his death was $250 000. It was invested at a rate of 6% for the two hundred years. If we know that each generation is separated from the next by about 25 years, with an average of three offspring per descendant, what share should each of the descendants receive? Assume that the last three generations are alive. This problem has been adapted from the will of Peter Fiddler, one of the early fur traders in Saskatchewan. (NUM)

---

1. Write a geometric sequence, and identify the first term, the common ratio, and the number of terms.

2. Find four geometric means between 20 and 1.25.

3. To what amount will an investment of $12 000 at a rate of interest of 5% compounded annually for 10 years, grow?

4. What amount must be invested today at a rate of interest of 8% compounded quarterly, for five years, to accumulate to a total of $3 575?

5. An annuity is purchased where the buyer pays $750 a year for each of 12 years. If the interest rate is 9% compounded yearly, how much is the annuity worth at maturity?

6. How much would a person have to invest each month for two years at 10% compounded monthly in order to accumulate $7500?

In addition, there are an adequate number of word problems in the resource texts available.

Students might be instructed to talk to financial representatives in their community to determine the meaning of the terms present value and annuity, and where they applied in that particular financial institution. Each student could give a brief description of the findings, orally or in writing.

Objectives	Instructional Notes
G.13 To determine the limit of a sequence.	The teacher can present the class with a set of sequences which have an infinite number of terms, and have students plot the first several terms of each sequence on a graph. This should enable students to determine visually whether the sequence has a limit (asymptote). Example. Plot each sequence on a graph. a) 1, 1/3, 1/9, 1/27, 1/81,....... b) 2, 3, 2, 3, 2, 3, 2, 3,............ c) 2, 4, 6, 8, 10,......... d) 1/3, 3/5, 5/7, 7/9, 9/11......... From this introduction, the teacher can then introduce the concept of a limit and the definitions of converging and diverging sequences. Some examples and exercises involving these concepts can be completed by the class.
G.14 To calculate the sum of an infinite series.	Students could be given one or two exercises in which to attempt to determine the sum of a given series. These might include the following; a) 9, 6, 4, 8/3,............ and b) 2, 3, 4.5, 6.75, ... Students should be able to identify a and r in both cases and substitute into one of the summation formulas. For the first, they should obtain $S_n = \frac{9[1-(2/3)^n]}{1-(2/3)}$, and $S_n = \frac{2[1-(3/2)^n]}{1-(3/2)}$ When they arrive at this point, they can be instructed to utilize their work with limits in the previous section to determine what happens to $(2/3)^n$ and $(3/2)^n$ as $n \to \infty$, and complete the question. Have them attempt one or two more exercises, and then determine the formula, using $S_\infty = \frac{a(1-r^n)}{1-r}$,first with $\|r\|<1$, and then $\|r\|>1$.

Examples/Activities

1. Determine whether each sequence converges or diverges. If it converges, supply the limiting value of the sequence. For this exercise, assume n must be a positive number.

   a) 2, 5, 8, 11, 14...........
   b) 6, 4, 8/3, 16/9, 32/27, ...
   c) 1/n, as n becomes increasingly larger.
   d) 1/n, as n becomes increasingly smaller, but remains positive.
   e) 3n, as n increases in value.
   f) $(2)^n$, as n increases in value.
   g) 2n/(2n+1), as n increases in value.

2. Determine the limit of each converging sequence. If a sequence does not converge, so state.

   a) $\lim_{n \to \infty} (6n-2)$
   b) $\lim_{n \to \infty} 3/(n+2)$
   c) $\lim_{n \to \infty} (3/4)^n$

Adaptations

The concept of limit is a topic that can be extended by the introduction of polynomials that are to be factored, the use of substitution, and choosing n to approach other values than ∞. Some examples are given below. In these, the teacher may also choose to instruct the students to determine the first three or four terms of the sequence.

Example 1.

Find $\lim_{x \to 5} \frac{3x^2-10x+3}{3x+1}$

Example 2.

Find $\lim_{n \to 0} \frac{3n+5}{2n}$

Example 3.

Find $\lim_{x \to -2} 4x^2 -5x -3$

---

1. Find the sum of each infinite series, if possible. If it is not possible, please explain why it is not.

   a) 8, 4, 2, 1,........
   b) 100, 90, 81, 72.9..........
   c) 24, -16, 32/3, -64/9,........
   d) a = 102, r = 1/6
   e) a = 48, r = -3/4
   f) 6, 8, 32/3, 128/9,......

2. Find the value of the indicated variable in each case.

   a) $S_\infty = 8$, a = 4, r = ?
   b) $S_\infty = 1/3$, r = 1/2, a = ?

3. Convert .363636.... to a rational number, using the sum of an infinite series.

4. A ball dropped from a height of 9 m rebounds 2/3 of its height on each successive bounce. How far does it travel before coming to rest? (Note that it travels both up and down, except for the first drop.)

This topic could be extended by employing a variety of word problems that can be taken from real-world situations. Many can be found using the available resource texts. (IL)

Example.

A journalist is given $200 to spend on a new form of entertainment that promises to return 90% of the money each time it is spent. The journalist finds that the money is eventually spent, but wants to know just what the total spent was. Calculate this total for the journalist.

Objectives	Instructional Notes
G.15 To solve word problems containing arithmetic or geometric series.	Students will need a brief review of the formulas for arithmetic series, as they were studied in Mathematics 10. It may be necessary to begin with one or two exercises in finding the sum of an arithmetic series before moving to word problems. Once the review is complete, students could work individually or in pairs on a set of word problems involving arithmetic and geometric series. The students should decide which type of series is inherent in the problem and what information is pertinent to the problem. Then the appropriate formula can be selected and the calculations carried out. Answers should be checked for reasonableness and solutions should be written.

Examples/Activities

1. Find the sum for each arithmetic series.

   a) $4 + 7 + 10 + 13 + \dots\dots + 40.$
   b) $a = 5, n = 25, t_n = 180, S_n = ?$

2. If the eighth term of an arithmetic sequence is 29 and the third term is 9, determine a and d, and the sum of the first twelve terms.

3. The sum of the first n terms of a geometric sequence is 21. If $a = 1/3$ and $r = 2$, what is n?

4. A warehouse employee stacks rows of boxes such that there are two fewer boxes in each successively higher row. If the bottom row has 57 boxes, and there are 23 rows, how many boxes are stacked together?

5. Yvonne is studying her genealogy. By using only parents, grandparents, great-grandparents, and other direct ancestors, she is able to trace her roots back 14 generations. Assuming she has traced all direct ancestors, how many different direct ancestors did she trace? (NUM)

Adaptations

Most of the text resources have word problems that are adequate for this section. You may wish to ask students to research some problems of these types on their own. (IL)

Objectives

G.16
To solve word problems involving compound interest or present value.

G.17
To solve word problems involving annuities or mortgages.

Instructional Notes

These next two sections could be done in conjunction with the previous section. These problems should be addressed throughout the unit and are shown as real-world applications of geometric sequences. The students could work individually, in pairs or in small groups to solve a few of these types of problems. Once the initial set has been solved, students could work individually on a short assignment of other problems of this type.

Various problems could also be posed or elicited from the students.

Decide on an amount of money you would like to earn as a yearly salary in the next few years. What amount of money would you need at a rate of 7% to generate this yearly salary? Call this amount m. Suppose you wish to retire at age 55. In order to have a retirement income equal to your salary, you need to save amount m. How much would you have to save each year, assuming a rate of interest of 7%, compounded yearly, to equal amount m?

Students should use calculators or computers to help with the calculations. They can also utilize logs in the calculations.

Examples/Activities

1. The Belczyk's are purchasing a home for $125 000. They find additional costs such as surveyors' certificates, lawyer's fees, transfer fees, title searches, and the like, bring the total cost to them to be $130 000. They are paying $25 000 down, and are to mortgage the remainder at 10.5% for 15 years, compounded monthly. What will their monthly payment be to pay off the mortgage in the specified time?

2. Twelve years ago, Kendra's parents began investing $750 a year at 8% interest compounded quarterly, for her education beyond high school. What is the value of this fund now? If they continued in this plan for another five years, what would be the value of the plan?

3. When she is 25 years of age, Kiena decides to plan for retirement. If Kiena invests $2 500 a year in an investment that averages 8% a year, compounded monthly, how much would the total investment be worth at age 50? age 55? age 60? If Kiena received 8% on her investment upon retirement, what monthly income could be expected at age 50 ? 55? 60?

Adaptations

Many financial institutions and businesses offer much information about RRSPs, mortgages, annuities, and similar investments. Much of this information is available free of charge in the form of pamphlets, which the students might read in order to understand better how compound interest affects their investment. In addition, employees of these institutions and businesses might be willing to speak to students about these concepts. (CCT) (COM) (IL) (NUM)

Mathematics C 30

Western Protocol - Common Curriculum Framework (1996)

10-12 Mathematics - General Outcomes

Number (Number Concepts)
- Analyze graphs or charts of given situations to derive specific information.
- Analyze the data in a table for trends, patterns and interrelationships.
- Explain and illustrate the structure and the interrelationship of the sets of numbers within the real number system.
- Explain and illustrate the structure of the complex number system and its subsets.

Number (Number Operations)
- Use basic arithmetic operations on real numbers to solve problems.
- Describe and apply arithmetic operations on tables to solve problems, using technology as required.
- Describe and apply arithmetic operations on matrices to solve problems, using technology as required.
- Make and justify financial decisions.

Patterns and Relations (Patterns)
- Represent naturally occurring discrete data, using linear or nonlinear functions.
- Generate and analyze number patterns.
- Investigate the nature of mathematical reasoning.
- Generate and analyze recursive and fractal patterns.

Patterns and Relations (Variables and Equations)
- Generalize operations on polynomials to include rational expressions.
- Represent and analyze situations that involve variables, expressions, equations and inequalities.
- Use linear programming to solve optimization models.
- Solve exponential, logarithmic and trigonometric equations.

Patterns and Relations (Relations and Functions)
- Examine the nature of relations with an emphasis on functions.
- Represent by models naturally-occurring data using linear functions.
- Represent and analyze functions using technology, as appropriate.
- Use the concept of function to solve problems.

Shape and Space (Measurement)
- Use measuring devices to make estimates and to perform calculations in solving problems.
- Demonstrate an understanding of scale factors, and their interrelationship with the dimensions of similar shapes and objects.
- Solve problems involving triangles, including those found in 3-D applications.
- Analyze objects, shapes, and processes to solve cost and design problems.

Shape and Space (3-D Objects and 2-D Shapes)
- Solve co-ordinate geometry problems involving lines and line segments.
- Develop and apply the geometric properties of circles and polygons to solve problems.
- Classify conic sections, using their shapes and equations.
- Solve problems involving triangles and vectors, including 3-D applications.

Shape and Space (Transformations)
- Perform, analyze and create transformations of functions and relations.

Statistics and Probability (Data Analysis)
- Describe, implement and analyze sampling procedures and draw appropriate inferences from the data collected, using mathematical and technical language.
- Apply line-fitting techniques to analyze experimental results.
- Analyze bivariate data.

Statistics and Probability (Chance and Uncertainty)
- Make and analyze decisions using expected gains and losses based on single events.
- Model the probability of a compound event in order to solve problems based on the combining of simpler probabilities.
- Solve problems based on the counting of sets, using techniques such as the fundamental counting principle, permutations and combinations.
- Use normal probability distribution to solve problems involving uncertainty.

Mathematics C 30

A. Mathematical Proof

Foundational Objective

- To appreciate the various types of mathematical thinking processes and to demonstrate skill in applying these processes (10 01 01). Supported by the following learning objectives.

B. Conic Sections

Foundational Objectives

- To become aware of the various conic sections, and to demonstrate skill in graphing and writing equations of the conic sections (10 02 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to solve systems of linear-quadratic and quadratic-quadratic equations (10 02 02). Supported by learning objectives 6 and 7.

C. Circular Functions

Foundational Objectives

- To demonstrate the understanding of trigonometric functions as developed by circular functions (10 03 01). Supported by learning objectives 1 to 5.
- To be able to produce the graphs of trigonometric functions (10 03 02). Supported by learning objectives 6 and 7.

D. Applications of Trigonometry

Foundational Objectives

- To demonstrate the ability to apply trigonometry to real-world problem situations (10 04 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to calculate areas of given triangles using trigonometry (10 04 02). Supported by learning objectives 6 and 7.

E. Trigonometric Identities

Foundational Objective

- To demonstrate the ability to work with trigonometric identities and to be able to apply them when necessary (10 05 01). Supported by the following learning objectives.

F. Trigonometric Equations

Foundational Objectives

- To demonstrate understanding and ability in solving trigonometric equations (10 06 01). Supported by the following learning objective.

Concept A: Mathematical Proof

Foundational Objective

- To appreciate the various types of mathematical thinking processes and to demonstrate skill in applying these processes (10 01 01). Supported by the following learning objectives.

Objectives

A.1
To define and illustrate by means of examples; deductive, inductive, and analogical statements or arguments.

Instructional Notes

Students should become familiar with types of arguments usually utilized in the real world. Examples of all types should be used to help the students determine the use and misuse of each. Many of these examples can be found in geometry texts but can also be located in other resources.

Students can be given a set of practice exercises to be discussed in small groups with the results to be shared with the class. A set of exercises to be done as homework could then be assigned. The students should not only be able to identify the type of thinking involved but also be prepared to comment on whether the result obtained is likely to be true for all cases. (COM)

Examples/Activities

A discussion of a problem, with justification, may be useful in setting up the need for proof.

Jolie travels from Saskatoon to North Battleford, a distance of 120 km. If she travels one direction at 100 kmh, and returns at 80 kmh, what is her average speed for the entire trip?

1. State which type of argument is being used in each case. Also, state whether the conclusion reached is valid.

   a) Ben was asked to complete the sequence 1, 1, 2, 3, . . . Ben wrote 5, 8, 11.

   b) Mary's mother stated that she was poor in writing, so therefore, Mary was likely to be poor in writing.

   c) Complete:
   All horses eat hay.
   Silver is a horse.
   Therefore, Silver ____________.

   d) Joan ate brunch at Cody's restaurant two Sundays in a row. She saw Brie there both times. Joan told her friends that Brie always eats brunch at Cody's.

   e) Vertically opposite angles are congruent. Angles A and B are vertically opposite. Conclusion:

Adaptations

Other types of problems can be utilized as examples. These may be diagrammatic, geometric, or from applied science.

E.g.: You have a litre of cola, and a litre of milk. You take a teaspoon (5 ml) of cola, add it to the milk, and stir. Then you take a teaspoon (5ml) from the milk container, add it to the cola, and stir. Which of the following is true: there is more cola in the milk container, more milk in the cola container, or equal amounts of both?

The teacher should also reinforce the principle of using a counterexample to disprove a statement. Effective counterexamples can help illustrate the need for proof and may help students to formulate arguments more clearly. (CCT)

One can also relate these types of thinking processes to other areas, such as in debating, parliamentary discussion, or the law.

Concept A: Mathematical Proof

Note: This objective has been covered in previous mathematics courses. Teachers may choose to use it as an optional topic at this level, if they feel that their students have retained most of their skills with this type of proof.

Objectives

A.2
To complete deductive proofs from geometry using a two-column format.

Instructional Notes

Students have had some experience in doing deductive proofs in Mathematics 20. The teacher may wish to use these types of proofs as an introduction to this section.

Once students have reviewed some of these, they might be introduced to other deductive proofs from geometry. These proofs could deal with congruent triangles, quadrilaterals, or involve some of the postulates dealing with circle geometry.

Students could initially work in small groups to discuss how they might approach a set of introductory proofs and to share the solutions of these introductory exercises. Once these have been done, the students could be given an assignment to be done on their own. Students may not have learned all the axioms, etc. required for a rigorous proof; therefore, some of their reasons may be written in sentence or paragraph form. (IL)

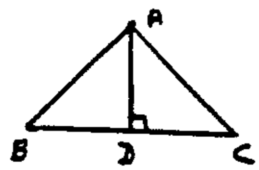
Examples/Activities

1. Given:

$\overline{AD} \perp \overline{BC}$, $\overline{BD} \cong \overline{CD}$

Prove:

$\overline{AB} \cong \overline{AC}$

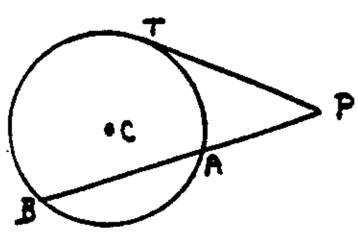


2. Prove that the diagonals of a rectangle are congruent.

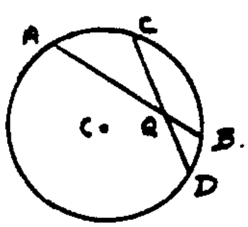
3. Prove that, in a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

4. Given: $\triangle$ ABC is isosceles. Prove the base angles are congruent.

5. Given: $\overline{PT}$ is tangent to circle C at T. $\overline{PB}$ is a secant.
Prove: $(PT)^2 = PA \cdot PB$



6. Given: $\overline{AB}$ and $\overline{CD}$ intersect at Q in circle O.
Prove: $AQ \cdot QB = CQ \cdot QD$



7. Prove that the diagonals of a parallelogram bisect each other.

Adaptations

As an alternative assignment, students could be asked to research geometry texts to locate some proofs that might be posed to the class. The student would be responsible for showing the correct proof of any question they pose. Students should offer to the class any problem they cannot solve and create a group learning opportunity.

Objectives	Instructional Notes
A.3 To complete proofs using some of the methods of co-ordinate geometry.	The teacher may wish to introduce this section by reviewing the distance formula, as it is employed in several types of coordinate proofs. The teacher can then pose a proof where students might make use of the methods of coordinate geometry. Many of the theorems involving length or equal measure lend themselves to this method. Some time should be taken to illustrate how one can set up the diagrams for these proofs, to use to advantage the x- and y-axes, and intercepts. Students should be given some proofs to practise in small groups, and then be given an assignment when they have seen how this system is used.
A.4 To use properties of numbers to justify solutions of alge-numeric exercises.	Using some basic algebra, students can be asked to demonstrate some properties of the number system. Students can work in small groups, pairs, or singly, to justify the statement or exercise given. Initial definitions such as the fact that an even integer can be written as 2n (where n is an integer) and an odd integer can be written as 2n + 1 (where n is an integer), can be demonstrated by the teacher to the class, or given to the class as a warmup exercise. Students can then be provided with a set of exercises to complete, or can be asked to research other mathematics resources for similar questions that they can be asked to present to the rest of the class.

Examples/Activities

1. Prove that the diagonals of a parallelogram bisect each other.
2. Prove that the diagonals of a rectangle are congruent.
3. Prove that the segment joining the midpoints of two sides of a triangle is equal to one half the length of the third side and is parallel to it.
4. Prove that the distance from each vertex of a right triangle to the midpoint of the hypotenuse is the same.
5. Prove that the medians of an equilateral triangle all have the same length.

Adaptations

Students could be asked to research mathematics texts for examples of coordinate proofs. They may be instructed to note the proof for one or two examples and to discuss these with the class. Some examples that might be found are those dealing with trigonometric identities. (IL)

Prove each of the following statements.

1. The sum of two even integers is even.
2. The difference between an odd integer and an even integer is odd.
3. The difference between two odd integers is even.
4. The sum of three consecutive odd integers is odd.

If there are a number of other resources, students could be asked to locate similar exercises and present them to the class in completed form as an example or pose them to the class as another exercise. (COM, IL)

Concept A: Mathematical Proof

Note: This objective can be dealt with in Concept E: Trigonometric Identities. It is placed here so the teacher can illustrate to students that the learning objective is one that employs skills in mathematical reasoning, but is more naturally taught in Concept E.

Objectives

A.5
To prove trigonometriconometric identities, using a two-column format.

Instructional Notes

Students can be asked to prove some basic trigonometric identities, such as those in Concept E. The teacher may wish to delay this section until trigonometric identities have been taught, or to do this section using some of the identities easily developed from triangle trigonometry from Mathematics 20 and Mathematics A 30.

In this section, it is expected that the students will verify specified identities and supply reasons for each step of the verification process. The students might begin by working in small groups, for purposes of peer tutoring and discussion, and then proceed to work on an assignment on their own.

Examples/Activities

Verify each of the following identities and supply reasons for each step in your validation. Be prepared to discuss your choice.

1. $\sin^2 \varnothing + \cos^2 \varnothing = 1$

2. $\sin a \cdot \cot a = \cos a$

3. $\dfrac{\sin^2 \varnothing + \cos^2 \varnothing}{\cos \varnothing} = \sec \varnothing$

4. $\sin \varnothing + \cos \varnothing \cot \varnothing = \csc \varnothing$

5. $1 - 2\sin^2 x = 2\cos^2 x - 1$

6. $\dfrac{\sin x\,(\csc^2 x - \cot^2 x)}{\cos x \sec x} = \sin x$

7. $(\tan a + \cot a)^2 = \sec^2 a + \csc^2 a$

Adaptations

Alternatively, the teacher could present some completed exercises and have the students supply the reasons for each step in each completed exercise.

Objectives

A.6
To introduce indirect proof and to use it in several proofs.

Instructional Notes

The introduction to indirect proof might be done by discussing a problem posed to the class. In this instance, we try to show a conjecture is false by providing a counterexample that leads to a contradiction. The initial examples chosen to present this to students should lend themselves to a contradiction that is relatively easy to visualize.

E.g.: The Prime Minister does not want to have Kuhn, Latta, and Moose dominate the Cabinet so he imposes some restrictions on their involvement. These are:

1. If Kuhn is a Cabinet member, then Latta is not.
2. If Kuhn is not a member, then Moose is.
3. If Latta is a member, then Moose is not.

Prove that Latta is not a member of Cabinet.

Some initial examples might also be taken from geometry, algebra, or from pure mathematics.

Once students have worked their way through several examples as a class, or in a small discussion groups, they can be assigned a set of exercises.

Examples/Activities

Use the method of indirect proof to prove each of the following assertions.

1. Prove that $\sqrt{2}$ is irrational.

2. Prove that a perpendicular drawn from a point to a line is the shortest distance from that point to the line.

3. Prove that two coplanar lines, both perpendicular to a third line, are parallel.

4. Prove that a collection of forty-three cent stamps and two-cent stamps that totals $1.39 must have an odd number of forty-three cent stamps.

When Tracey went to the store the score was Edmonton 21, Saskatchewan 20 with 2:19 left to play. The final score was Edmonton 21, Saskatchewan 22. The announcer said that this game was unusual because no single point scores were made (except for the points after touchdowns). Tracey concluded that the game was won by the Saskatchewan defence scoring a safety (two points). Use indirect proof to verify Tracey's conclusion.

Adaptations

The history of indirect proof can be researched by the students and short reports can be given to the class. The students might also be asked to locate instances where the indirect method of proof is utilized in resource materials and to present some of these instances to class. (IL)

Examples of indirect proof can be found in most geometry texts but look for other examples in algebra texts as well. The examples should not be limited to geometry.

Objectives

A.7
To introduce the principle of mathematical induction.

Instructional Notes

The principle of mathematical induction is a two-stage process. First, a statement S must be shown to be true for the case S(n), where n = 1. Then show that the assumption that the statement is true for the case n = r implies that it is also true for the case n = r + 1; that is, show that the hypothesis S(r) implies S(r + 1).

This process is usually used when proving statements involving natural numbers and when we have a set of statements to prove.

Initial examples should be chosen for class presentation or discussion so that students can internalize the process.

One of the most common examples is to determine the sum of the natural numbers

$$1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n + 1)}{2}$$

1. The statement is true for n = 1, since by substitution, both the left and right sides equal 1.
2. Assume $1 + 2 + 3 + \ldots + r = \frac{r(r + 1)}{2}$
3. Required to prove: $1 + 2 + 3 + \ldots +(r + 1) = \frac{(r + 1)\,[(r + 1) + 1]}{2}$

Proof

$$1 + 2 + 3 + \ldots + r + (r + 1)$$

$$= \frac{r(r + 1)}{2} + (r + 1)$$

$$= \frac{r(r + 1)}{2} + \frac{2(r + 1)}{2}$$

$$= \frac{(r + 1)(r + 2)}{2} = \frac{(r + 1)[(r + 1) + 1]}{2}$$

Examples/Activities

The intent of the exercises in this section is to allow students to obtain practice in constructing proofs using mathematical induction.

1. Find S_{r+1} in each case.

   a) $S_r = \frac{r(r-1)^2}{6}$

   b) $S_r = (3r + 2 - 4r^2) + (3r - 5)$

   c) $S_r = 2 + 5 + 8 + \ldots + (5r - 1) + (5r + 2)$

   d) $S_r = (r^2 - 3)\,[(r + 1)^2 - 3]$

Adaptations

Many of the newer resource texts have a section or two on mathematical induction. Students can be assigned exercises from these texts and asked to share their proofs with the class.

Concept A: Mathematical Proof

Objectives

A.8
To prove assertions using mathematical induction.

Instructional Notes

In this section, students should be able to use the principle of mathematical induction to construct proofs. These proofs will most likely be drawn from their work on sequences and will utilize the process as outlined in the previous section. The teacher may wish to model a few examples for the students, or have them work through some examples in their discussion groups.

Once the students have had some practice in doing these proofs, it is expected that they will be asked to complete several of these proofs on their own.

Examples/Activities

Prove each of the following statements.

1. Prove that the sum of the first n positive even integers is $n^2 + n$.

2. Prove that $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$.

3. Prove that the sum of the squares of the first n natural numbers is given by
$$\frac{n(n + 1)(2n + 1)}{6}$$

4. Prove that $1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}$

5. Prove that the sum of the cubes of the first n natural numbers is given by
$$\frac{n^2(n + 1)^2}{4}$$

6. Prove that
$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

7. Prove that $x^2 + 5x$ is an even number for any x which is a positive integer.

8. Prove, by mathematical induction
$$\sum_{k=1}^{k} (2k) = k(k + 1)$$

9. Prove that for $x > 0$, and where n is a positive integer, that $(1 + x)^n > nx$

Adaptations

Students could be given exercises in which they would be asked to use mathematical induction to illustrate some other mathematical statements other than sums.

Some examples of these might be:

1. Use mathematical induction to prove the following exponential 'law'.

   $(xy)^m = x^m y^m$

2. Use mathematical induction to prove that $(3/4)^n < n$, for $n \geq 0$.

Concept B: Conic Sections

Foundational Objectives:
- To become aware of the various conic sections and to demonstrate skill in graphing and writing equations of the conic sections (10 02 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to solve systems of linear-quadratic and quadratic-quadratic equations. (10 02 02). Supported by learning objectives 6 and 7.

Note: This development of conics is parallel to that proposed by the Western Protocol Curriculum Framework (1996) and is all that is required for C 30. However, if teachers find they have time available, a more traditional approach to Conic Sections is developed in Appendix F of this document.

Concept B: Conic Sections

Objectives/Skills	Instructional Notes
B.1.a) To convert the equation of a circle from the general form to the standard form and vice versa.	Students may need to be reminded of the process of completing the square and given a few warmup exercises using this process.
B.1.b) To sketch the graph of a circle.	The students could be instructed to work in small groups, pairs, or singly, to convert these equations from one form to the other. The general form of a conic section is considered by most to be $Ax^2 + By^2 + Cx + Dy + E = 0$ (Note: Do not use an xy term at this level), while the standard form of a circle is $(x - h)^2 + (y - k)^2 = r^2$. When students have had some practice in converting these types of equations, the teacher can quickly review some of the basic definitions of circles and have the students sketch the graphs of these circles, using the centre and radius.
B.2.a) To convert the equation of a parabola from the general form to the standard form and vice versa.	The teacher may wish to introduce this section by reviewing the graphing of parabolas as done in Mathematics 20 and having the students identify the key characteristics of a parabola. Then, the students can be given a short set of exercises to work on in small groups, pairs, or individually, in order to have some practice in converting equations from one form to the other. (PSVS)
B.2.b) To sketch the graph of a parabola.	

Examples/Activities

1. Convert each of the following to general form.
   a) $(x-3)^2 + y^2 = 16$
   b) $(x + 2)^2 + (y -3)^2 = 25$
   c) $(x + 5)^2 + (y - 1)^2 = 4$

2. Convert each of the following to standard form.
   a) $x^2 + y^2 + 4x + 6y = 12$
   b) $x^2 + y^2 - 6x - 2y = 6$
   c) $3x^2 + 3y^2 + 6x - 6y = 42$
   d) $x^2 + y^2 + 8x - 6y + 30 = 0$

3. Sketch each of the circles as described in the questions above, if possible.

Adaptations

The teacher may wish to use technology for this section, and have the students complete the graphing and analysis using this technology. (TL)

Another adaptation is to introduce elements of problem solving, either by introducing real-world problems from various resources, or supplying a few clues so that the students have to identify the centre and radius in order to construct the equation. An example might be: the endpoints of a diameter of a circle are given by (-2,0) and (4,0). Sketch the graph of the circle and determine its equation.

---

1. Sketch each of the following and convert to general form.

   a) $y = (x - 2)^2 + 3$
   b) $y = 2(x + 1)^2 - 1$
   c) $y = 4(x + 3)^2 + 11$
   d) $y - 4 = 1/2(x -2)^2$
   e) $x + 3 = 2(y - 1)^2$

2. Convert each of the following to standard form and then sketch the graph of each.

   a) $x^2 - 2x - y = 3$
   b) $x^2 + 6x + y = -5$
   c) $3x^2 - 12x + 16 = y$

The teacher may wish to use technology to graph any or all of the equations and to have the students utilize the technology to do the analysis of each graph. (TL)

Real-world problems should also be introduced wherever possible. Students can be given a set of these problems to work at, or instructed to research resources to find examples of related real-world problems.

Concept B: Conic Sections

Objectives/Skills

B.3.a)
To convert the equation of an ellipse from the general form to the standard form and vice versa.

B.3.b)
To sketch the graph of an ellipse.

Instructional Notes

In this course, it is necessary only to deal with ellipses that have vertical and horizontal axes.

The teacher may wish to introduce this section by defining an ellipse and some of the characteristics associated with the ellipse such as the vertices and axes.

Students can be given a short set of exercises to work at in small groups, pairs, or singly, in order to practise converting from one form to the other.

Examples/Activities

1. Sketch the graph of each equation and then convert the equation to general form.

   a)

   $$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

   b)

   $$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

   c)

   $$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{9} = 1$$

2. Convert each of the following equations to standard form and then sketch the graph of each equation.
   a) $25x^2 + 4y^2 = 100$
   b) $4x^2 + 9y^2 + 8x = 32$
   c) $25x^2 + 16y^2 - 150x - 64y = 111$

Adaptations

Technology can be used to graph and analyze the ellipse. The teacher may wish to have students develop the concept of an ellipse and its characteristics through an exploratory approach, using technology. (TL)

The introduction of real-world problems should also be considered; either supplied by the teacher or researched by the students.

Concept B: Conic Sections

Objectives/Skills

B.4.a)
To convert the equation of a hyperbola from the general form to the standard form and vice versa.

B.4.b)
To sketch the graph of a hyperbola.

Instructional Notes

The teacher could introduce this section by illustrating a hyperbola, defining it, and highlighting its major characteristics. These would include the vertices and asymptotes.

Students could be given a short set of exercises to enable them to practise converting the equation from one form to another.

The teacher might wish to demonstrate to the students how to sketch the graph of the hyperbola using the vertices and asymptotes.

As an alternative, the graphing could be done using technology, in an exploratory fashion. (TL)

Examples/Activities

1. Sketch the graph of each hyperbola and then convert each equation to general form.

   a)

   $$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

   b)

   $$\frac{(x+3)^2}{9} - \frac{(y-2)^2}{4} = 1$$

   c)

   $$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

2. Convert each of the following equations to standard form and then sketch the graph of each hyperbola.

   a) $9x^2 - 4y^2 = 36$
   b) $25x^2 - 4y^2 + 100x - 8y = 4$
   c) $9y^2 - 16x^2 - 54y = 63$

Adaptations

The introduction of real-world problems, either supplied by the teacher or researched by the students, is a way of extending the practical knowledge of a hyperbola.

A research topic on navigation using LORAN (LOng RAnge Navigation) might be an interesting project for motivated students.

At this point, relating the circle, parabola, ellipse, and hyperbola to the double-napped cone might serve as a good summary.

Concept B: Conic Sections

Objectives/Skills

B.5
To examine the coefficients of the second degree equation $Ax^2 + By^2 + Cx + Dy + E = 0$ and to identify the conic section it represents.

Instructional Notes

Once the students have studied the conic sections, they should be able to identify the type of conic section represented by the equation. This can be done by focussing on the coefficients of Ax^2 and By^2 in the general equation
$Ax^2 + By^2 + Cx + Dy + E = 0$.
Case I: $A = 0$ and $B = 0$, -- a line
Case II: $A = 0$, B = real no. --parabola
A = real no., $B = 0$ --parabola
Case III: A and B = same real no.-- circle
Case IV: A,B = different real nos., same sign -- ellipse
Case V: A,B = real nos., signs different, -- hyperbola

Students could be given a series of equations to examine and state the type of conic section each one represents, by observing these coefficients.

B.6
To sketch diagrams to show possible relationships and intersections of the following systems: Linear-Quadratic and Quadratic-Quadratic.

Once students are able to identify the type of conic by inspecting the coefficients of an equation, as in the previous objective, they can be given systems of equations to identify and asked to speculate on the number of points of intersection that are possible for each system. After their initial guess, they might be asked to generate the sketches of the pair of equations of the system, in order to check their result.

As an alternative, they can also be asked to draw sketches of systems that illustrate a different number of points of intersection. As an example, they could be asked to sketch a hyperbola and an ellipse that have four points of intersection, three points of intersection, two points of intersection, one point of intersection, or no points of intersection. This type of exercise can be repeated for any pair of conic sections, with the exception of those systems that include one linear equation.

Examples/Activities

Given the equations below, identify which type of conic section each represents.

1. $6x^2 + 12x + 6y^2 - 8y = 100$
2. $4x^2 + 12x + y^2 - 6y = 64$
3. $9x - 6y = 7$
4. $6y = 3x^2 - 10$
5. $2x^2 + 8x - 25 = y^2 + 4y$
6. $-3x^2 - 6x + 8y = 15$
7. $4y^2 - 8y + 6x^2 + 12x = 36$
8. $9y^2 - 18y - 4x^2 + 8x = 121$
9. $3x^2 + 6x + 3y^2 - 12y = 49$
10. $5x^2 + 15x + 3y^2 + 15y = 81$
11. $4y^2 - 12y + 8x = 24$
12. $-5x^2 + 10x + 5y^2 - 10y = 25$
13. $8x - 3y = 24$
14. $4x^2 - 3y^2 + 16x - 9y = 49$
15. $-7x^2 - 14x + 25 = 7y^2 + 21y$
16. $4xy = -12$

For each of the following systems, identify the conic represented by each equation and speculate on the number of possible intersecting points. Sketch each system on the same coordinate axes to determine if your original answer was correct.

1. $x^2 + y^2 = 25$
   $x^2 - y^2 = 25$
2. $3x + 2y = 6$
   $4x^2 + y^2 = 36$
3. $4y = 3x^2$
   $3x^2 - y^2 = 9$
4. The path of a comet is roughly elliptical. Scientists determine that the orbit of a particular comet is a danger to the Earth and therefore it must be destroyed. They agree that the method of destruction can only be a nuclear warhead that will travel in a straight line once it breaks away from the Earth's gravitational field. Draw a sketch illustrating the comet and the warhead and show the number of possible 'points of destruction'. (CCT)

Illustrate the various number of points of intersection possible for each of the following systems of equations.

1. A parabola and an ellipse
2. A circle and a line
3. A hyperbola and a parabola
4. A circle and a hyperbola
5. A line and a parabola
6. An ellipse and a parabola

Adaptations

The teacher may wish to have students identify conics in this manner and then have them work through some to see if they are the conics the students predicted. Including one or two cases where no solution exists could lead to a further discussion of properties and reinforce the fact that this is but one screening device.

E.g.:
$x^2 + y^2 - 6x + 4y + 4 = 0$

has an equation that students may be tempted to answer represents a circle. However, one can convert it to standard form and realize it does not have a radius and cannot be a circle.

As an adaptation of this exercise, students might be asked to graph each pair of equations in the system and be asked to identify the coordinates of the intersecting points. This would be solving the system by graphing.

Concept B: Conic Sections

Objectives/Skills	Instructional Notes
B.7 To solve the following systems of equations algebraically. Linear-Quadratic and Quadratic-Quadratic.	Students can be given a set of systems to solve algebraically using the methods of comparison, elimination, or substitution. The teacher may wish to review these methods as learned in the section on solving systems of equations of lines in Mathematics A 30. The students should be instructed to examine the answers to obtain all possible solutions. (For example, if $x^2 = 9$, both 3 and -3 might be solutions for x, and then used to find y - values.) Students should also be instructed to check all possible solutions to determine exactly which ones form the solution set.

Examples/Activities

Solve each system of equations algebraically.

1. $x^2 + y^2 = 9$
   $9x^2 - 4y^2 = 36$

2. $x^2 + 4y^2 = 16$
   $x + 2y + 4 = 0$

3. $x^2 + y^2 = 100$
   $2x - y = 8$

4. $4x^2 - y^2 = 7$
   $2x^2 + 5y^2 = 9$

5. $y = x^2 + 4$
   $x + y = 16$

6. $4x^2 + y^2 = 36$
   $x^2 + y^2 = 16$

7. $x^2 + y^2 = 25$
   $x^2 - y^2 = 7$

8. $x^2 - y^2 = 5$
   $2x - y = 4$

9. $x^2 - y^2 = 9$
   $x^2 + y^2 = 9$

10. $x^2 + y^2 = 13$
    $xy = 6$

* Use substitution and factoring to solve.

Adaptations

As an adaptation to this section, students could be paired, with one graphing each system and determining the coordinates of intersecting points and the other solving algebraically.

The students in each pair would switch roles halfway through the assignment. They could compare the solutions each got using his/her respective method in each question, and discuss the results. E.g.: Which method is more effective? Which gives a quicker explanation? Where might each be used?

Concept C: Circular Functions

Foundational Objectives:

- To demonstrate an understanding of trigonometric functions as developed by circular functions (10 03 01). Supported by learning objectives 1 to 5.
- To be able to produce graphs of trigonometric functions (10 03 02). Supported by learning objectives 6 and 7.

Objectives/Skills	Instructional Notes
C.1 To define the trigonometric functions and real numbers by wrapping a number line around a circle.	Using a unit circle and a number line tangent to the unit circle at (1,0), the number line can be visualized as being wrapped around the circle. Since the circumference of the unit circle is 2π, the distance around the circle to (-1,0) is π. Each point on the circumference (x,y) can thus be related to a real number (the length of the arc) described in terms of π or as a real number in the same units as the original number line. Since it is a unit circle, the radius, r, is always 1, and the trigonometric functions of any angle ϕ in standard position can be found by using x, y, and r. Since r is 1, the sine of any point is $y/r = y/1 = y$. Similarly, the cosine is x, and so on. Also make sure that students have several opportunities to observe that angles of more than 2π are circular and give the same result as angles $\pm 2\pi$ different in size.
C.2 To determine values of the primary and reciprocal trigonometric ratios.	Students should have the opportunity to practice determining the trigonometric functions of any angle ϕ in standard position in a unit circle. Also have students practise determining the trigonometric functions of angles $\geq 2\pi$ or of angles $\leq -2\pi$. **Note:** When dealing with the reciprocal functions of cosecant, secant, and cotangent, it is recommended that any graphs illustrating these be kept at a basic level.

Examples/Activities

In which quadrant is each of the following angles located?

1. $\pi/3$	5. $8\pi/5$
2. $5\pi/3$	6. $-4\pi/3$
3. $-\pi/4$	7. $6\pi/5$
4. $7\pi/$	8. $-3\pi/4$

What ordered pair is associated with each of the following values obtained from wrapping a number line around a circle?

1. 6π	6. $10\pi/4$
2. -3π	7. $-\pi/2$
3. $-3\pi/2$	8. -10π
4. $3\pi/2$	9. $17\pi/2$
5. 9π	

Given a point lying on a unit circle, determine the six trigonometric functions for an angle in standard position whose terminal side contains the given point.

1. (1,0)	4. (.8660,-.5)
2. (-.5,.8660)	5. (0,1)
3. (-.7071,-.7071)	6. (-1,0)

Find (x,y) given the length of the arc of the circle in each of the following. Determine the six trigonometric functions in each case.

1. $\pi/2$	5. $3\pi/2$
2. $\pi/4$	6. $3\pi/4$
3. $\pi/3$	7. $-3\pi/4$
4. $\pi/6$	

Adaptations

Students could be asked to construct a unit circle and a number line with the same units and illustrate the wrapping technique. They could be asked to identify specific points (e.g.: $\pi/2$) on their number line and determine the coordinates. Based on their construction and measurements, they could then be asked to determine the six trigonometric functions of some specific points.

Objectives/Skills	Instructional Notes
C.3 To determine the radian measures of angles, to convert from radians to degrees and vice versa.	Students will be familiar with the concept of a unit circle by this point and should readily be able to contribute the fact that 180° is the equivalent of π units from the wrapping function. At this time, the introduction of radians as the unit of measurement for our wrapping function is recommended. The conversion from radians to degrees and vice versa can be done using the fact that $180° = \pi^R$. Simply use this as a basis of setting up a proportion with one unknown quantity, and solve. From this ratio, the teacher may wish to develop specific conversions for each type, but this is not necessary. Example. Convert 45° to radian measure. Solution. $\frac{180^0}{45°} = \frac{\pi^R}{?^R}$ Now solve the proportion for$?^R$. Your answer is $\pi/4^R$. Alternatively, use the standard measures as the denominators to make the solution of the proportion easier. In our example, the original proportion would be written $\frac{45°}{180°} = \frac{?^R}{\pi^R}$
C.4 To determine angular velocity and to apply this concept to solving problems involving rotation.	The concept of velocity could be introduced as v = d/t, where v is the velocity, d is the distance travelled, and t is the time. Note that this formula requires that d and t use the same measures of distance. This can be adapted to angular velocity by replacing the distance d with ϕ, the radian measure of the angle which corresponds to the arc described. ϕ is used to describe the path of a particle or point in angular velocity. Once students have been introduced to this variation of the standard formula, they can be given some problems to solve on their own, or in small groups. These can be corrected as a class, and any difficulties solved. Once students have had the opportunity to do a few of these problems, a short assignment can be given.

Examples/Activities

1. Convert to radian measure:
   a) 60° b) 390° c) -135° d) 144° e) 75°

2. Convert to degree measure:
   a) $3\pi/2^R$ b) $5\pi/8^R$ c) $-5\pi/3^R$ d) $5\pi/6^R$ e) 2^R

3. Determine the six trigonometric values for each of the following angles.
   a) $\pi/6^R$ b) $\pi/4^R$ c) $\pi/3^R$ d) $3\pi/2^R$ e) $5\pi/4^R$ f) $7\pi/3^R$ g) $11\pi/6^R$

Adaptations

Students should be given some time to work with the radian and degree modes of their calculators, in order to become familiar with the operation of the calculator they are using. The correct usage of the radian mode in this section can reinforce the concepts learned in class. (TL)

---

1. The second hand of a clock is 20cm long. Find the velocity of the tip of the second hand. (This question could be repeated using the minute hand.)

2. A Ferris Wheel at an amusement park has a radius of 8m. If it completes two revolutions in 11 seconds, determine the angular velocity of the Ferris Wheel in radians per second.

3. A bicycle has a tire with a radius of 65 cm. What is the velocity of the tip of one spoke, if Ila is riding such that the tire is revolving at 90 revolutions per minute? What is the angular velocity of the tire?

4. The moon is approximately 3.86×10^5 km from the Earth. It completes one revolution about the Earth every 28 days. How far does it travel in one day? What is its angular velocity for one day?

Students may be asked to propose other real-world situations in which angular velocity might be calculated. These could be based on their own experiences. If time permits, some modelling of these situations, or explanation and solving a related problem might be considered.

These situations might involve tires, fishing reels, saws, pulleys, microwave turntables, barbecue spits, cassette tapes, and the like. (TL)

Concept C: Circular Functions

Objectives/Skills

C.5
To determine arc length and to apply this in associated problems.

Instructional Notes

Students should become familiar with the terms associated with the determination of arc length. The concept of arc length may have to be modelled or explained, in order that the students appreciate what they will be asked to determine. The variables s, r, and ϕ^R must be introduced and students should be given some opportunity to familiarize themselves with these.

Students might try to measure the arc length of a part of a given circle by traditional means (tape measures, string, etc.) and then asked to employ some mathematical method (portion of the circumference of a circle). After this, they could be instructed to use the formula for arc length, $s = r\phi^R$, and to compare the results obtained from the various trials.

Students can then be given a set of exercises that will allow them to determine arc lengths in various circumstances.

Examples/Activities

1. Determine the value of the indicated variable (?) in each case: (The teacher may require the answer left in radians, to two decimal places, or other measures).
   a) $s = ?$, $r = 10$ cm, $\phi = \pi/3^R$

   b) $s = 27$ cm, $r = ?$, $\phi = 5\pi/4^R$

   c) $s = 35$ cm, $r = 17.5$ cm, $\phi = ?$

   d) $s = 2.1$ m, $r = ?$, $\phi = 75°$

2. Determine the arc length subtended by a radius of 5 cm and an angle of $\pi/3^R$.

3. Determine the central angle if the radius of the circle is 1.25 m and the arc length is 10.5 m.

4. Determine the arc length defined by circle of radius 31.5 cm and a central angle of 120°.

5. If the Earth is assumed to be a sphere of radius 6 400 km, and the United States-Canada border is on the 49th parallel (49° North of the Equator), what is the latitude of some of the larger centres in Saskatchewan; such as, Regina, Saskatoon, Prince Albert, La Ronge, your hometown, etc? Use a highway map and its scale to determine how far north of the border each is and then determine the latitude. This problem can be redesigned for longitude, in order to explore how many km in one degree of longitude, how many km per time zone, calculation of the speed of rotation of the Earth, and so on.

Adaptations

Many real-world problems can be found to illustrate arc length and the previous topic, angular velocity. If the teacher does not have time and the students are unable to generate many of their own, the teacher might have the students research these types of problems from other resource texts available in the school, or from other sources in the Resource Centre or community.

The employment of a surveyors' wheel for measuring distances may help some students gain a greater appreciation of mathematics in the real world.

Concept C: Circular Functions

Objectives/Skills

C.6
To define and illustrate the following terms: periodic function, amplitude, domain, range, minimum value, maximum value, translation, wave motion, sinusoidal functions.

Instructional Notes

The basic trigonometric function graphs can be utilized here. Students should be able to graph each of the six basic trigonometric functions from values, using calculators, computers, or researching various texts.

Students can be instructed to add all necessary terms to a glossary, to define and illustrate each, to complete a chart showing the definition and an illustration for each term, or some other method which will ensure the student has a demonstrated degree of familiarity with these terms.

As an exercise, the student could be given a set of graphs, and asked to identify specific terms for each graph: what is the range, domain, maximum and minimum values, amplitude, etc? Does it represent a periodic function? The students could then be asked to translate the given graph to a new set of axes and to determine which of the previous values change because of the translation. Which remain the same?

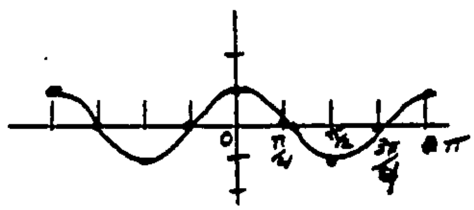
Examples/Activities

1. Complete the following chart:

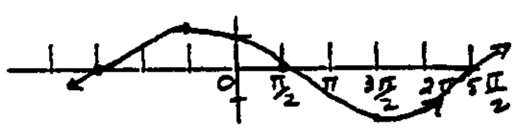
Name	Definition	Illustration
domain		
range		
amplitude		
max. value		
min. value		
period (w.l.)		
translation		

2. Given the following graphs, identify the amplitude, wave length, and maximum and minimum values.

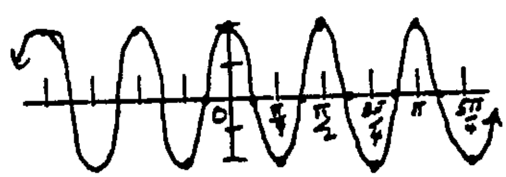
a)



b)



c)



3. For a), b), and c) of question 2, sketch the graph given :
   a) a phase shift of $\pi/4$
   b) a vertical shift of -2
   c) a phase shift of $-\pi/3$, and a vertical shift of 3.

4. Sketch the graphs of sin ϕ, cos ϕ, tan ϕ, csc ϕ, sec ϕ, and cot ϕ. Which of these are functions? periodic functions? sinusoidal functions? (Hint: Use your graphic calculator, a computer program, or resource text to help you sketch these.)

Adaptations

Students can be asked to identify x- and y- intercepts of these graphs, and speculate on the usefulness of these. This can foreshadow the section on solving trigonometric equations.

Objectives/Skills

C.7
To state the range, period, amplitude, phase shift, minimum and maximum values and to sketch the graphs of:

a) $y - k = a \sin(x - h)$
b) $y - k = a \cos(x - h)$
c) $y - k = a \tan(x - h)$

Instructional Notes

Once students have demonstrated an understanding of the previous section, they can be asked to analyze and sketch graphs given the equations. They should realize that the basic shapes remain constant and should be able to identify the items stated in this objective.

An assignment in this section might consist of a series of equations for which the students are expected to graph and analyze each equation.

Note: Any graphs involving the reciprocal trigonometric functions should be kept at a relatively basic level.

Examples/Activities

1. Sketch the graph of each equation. Identify the range, period, amplitude, maximum and/or minimum values, and phase shift for each, if they exist.

   a) $y = \sin(x + \pi/2)$

   b) $y = 3\cos(x - \pi/2)$

   c) $y = -2 + \cos 3(\phi + \pi/2)$

   d) $y = 4/3 - 2\sin(5\phi + \pi)$

   e) $y = 2\tan\phi$

   f) $y = 3/4 \tan(\phi - \pi/3)$

   g) $y = \csc\phi + 2$

   h) $y = \sec\theta - 1$

2. Write an equation for a sine function, given that

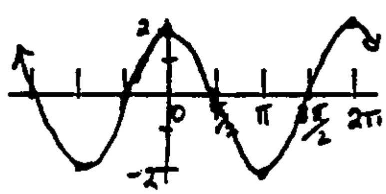
   a) Amp. = 2, period = $2\pi/5$

   b) Vertical shift = -2, amp. = 3, period = $\pi/2$, phase shift = $-\pi/4$

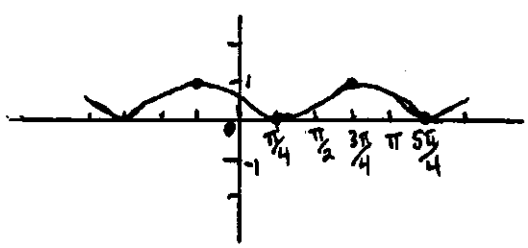
   c) Max. = 5, min. = 1, period = 4π, phase shift = $\pi/3$

3. Given the following graphs, complete the analysis (vertical shift, phase shift, amplitude, and period) and write the equation of the graph.

   a)



   b)



Adaptations

Students can be asked to note the graphs of functions similar to $y = \sin(\phi - \pi/2)$ and $y = \cos\phi$. After examining several pairs of such related graphs, the students may be asked to write down some observations and conjectures. A class discussion might be utilized to determine any conclusions. The students could then verify or reject any such conclusions by completing further examples.

This can be extended to other pairs of co-functions, if the students do not raise this issue on their own. E.g.: Does this also work for graphs of functions such as $y = \tan(\phi - \pi/2)$ and $y = \cot\phi$? Is the phase shift the same for all pairs? Does it vary? Why?

An activity that should be done by all students is the solving of related word problems. These may be found in most resource texts. Although there are not very many problems, the ones found should be utilized if at all appropriate. A common theme is the rise and fall of ocean tides. (IL)

Concept D: Applications of Trigonometry

Foundational Objectives:

- To demonstrate the ability to apply trigonometry to real-world problem situations (10 04 01). Supported by learning objectives 1 to 5.
- To demonstrate the ability to calculate areas of given triangles using trigonometry (10 04 02). Supported by learning objectives 6 and 7.

Objectives/Skills

D.1
To define and illustrate the following terms: angles of elevation and depression, heading, bearing, compass direction.

Instructional Notes

An introduction to this topic might begin with a brief review of the solution of right triangles, using the sin, cos, and tan functions. Examples and illustrations of the terms angles of elevation and depression could be used and students could be asked for real-world examples that would illustrate the correct use of these terms.

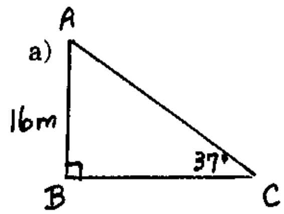
The terms associated with direction might be studied using maps, actual compasses, or through an assignment based on a 'treasure hunt', where six or seven instructions are given in terms such as "N 35° E, 7 cm" or "bearing 265°, 8 cm". These instructions can be provided with a map and a starting point, and students can be asked to determine where the "treasure" is.

Once students are familiar with the definitions associated with this section, they might be asked to draw diagrams illustrating the situation described in given situations.

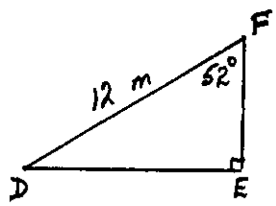
Examples/Activities

1. Solve each of the following right triangles:

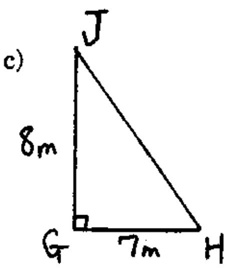
a)



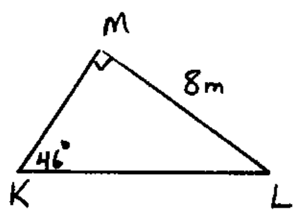
b)



c)



d)



2. Draw a diagram that represents each of the following situations:

   a) A student at the top of a knoll looks down at a village in the valley. The knoll is known to be 30 m in height, and the student measures the angle of depression to be 25° from the top of the knoll.

   b) A bookkeeper is sitting at a desk on the third floor of a building. The building across the street has eight floors. Illustrate a line of sight from the bookkeeper to the seventh and second floors of the building opposite and label the angles of elevation and depression.

Adaptations

Students can be given a map and a take-home assignment where they create a series of steps that another student (the solver) must follow in order to reach a designated point on the map.

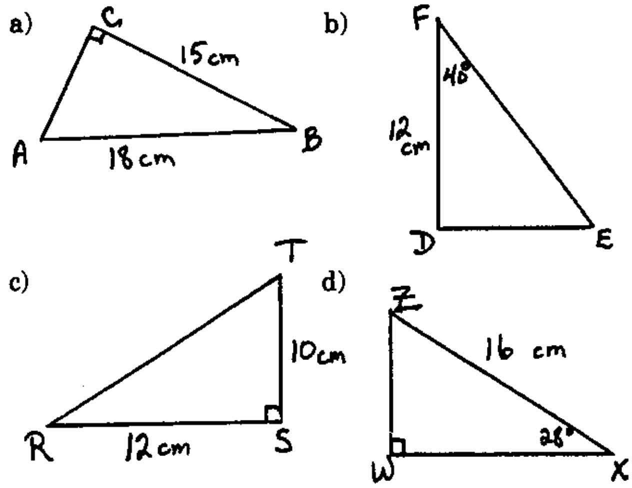
The solver can then take the problem for homework the next day. If the correct point is not obtained using the information, both the author of the problem and the solver must decide where the error occurred. The steps should be written in terms of the definitions needed for this section. (IL, PSVS)

Objectives/Skills	Instructional Notes
D.2 To solve right triangles and associated word problems.	In this section, students will be expected to set up and solve problems associated with right triangles. These include problems based on the terms studied in the previous section. The teacher can begin with the solutions of right triangles and then introduce problems after a few examples have been worked by students. In solving right triangles, students could be asked to use the reciprocal functions in cases where the unknown quantity is in the denominator.

Examples/Activities

Adaptations

1. Solve each of the following triangles:



2. A plane is flying on a heading of 160° and at a speed of 400 kmh. After two hours, it changes course to a heading of 250°, and continues flying. How far is it from the airport, and what is its heading in relation to the airport, after one hour on its corrected flight path?

3. Nikita's kite is flying at the end of her string, which is 100 m long. She estimates the angle of elevation to the kite is 35°. How high is the kite flying, given this estimate?

4. Kendra spots a forest fire from a lookout tower 30 m high. She determines the angle of depression to the fire from the tower to be 2°. How far from the base of the tower is the fire? in meters? kilometres? Which is a better unit of measure?

5. A student is told that the height of the flagpole on top of a building is 3 m. From a point 100 m from the building, the angle of elevations to the top and bottom of the flag pole are 54° and 42° respectively. How does the student determine the height of the building?

Concept D: Applications of Trigonometry

Objectives/Skills	Instructional Notes
D.3 To solve oblique triangles by the use of the Law of Sines/Cosines.	The Law of Sines and the Law of Cosines must be introduced to the class. These can be developed by the teacher, or by the class guided by the teacher. Most of the resource texts have a derivation of these laws and illustrate how they can be derived. The students should obtain some practice in using these laws with oblique triangles that have been carefully chosen to have one unique solution. The first examples should instruct the student to determine all the known information and to identify the parts of the triangle to be solved. The student should also decide which of the two laws to employ in the first instance. Examples should be chosen so that students must make different decisions in each question.
D.4 To solve triangles including all solutions given two sides and a non-included angle (the Ambiguous Case).	To begin this section, the teacher may wish to have students find ϕ for any $\sin \phi$ = a positive value. Students should be able to determine two values for ϕ in this case. Further, the teacher can continue with $\cos \phi$ = a positive value, and determine two values of ϕ in each case. It might be pointed out to students that while both values of ϕ are possible for the case $\sin \phi$ (because each value is less than 180°), the same is not true for $\cos \phi$. The teacher may wish to illustrate an example or two involving the ambiguous case. The students might be better able to predict when to expect the ambiguous case, if they compare the given triangle information to the triangle congruence postulates. For information which parallels SSS, SAS, AAS, or ASA, one unique solution can be found. In addition, for AAA, no solution can be found, and the case SSA is the only one where one has to check for the ambiguous case. If the students are able to predict these patterns before they begin the solution, they will know what to expect. In cases where the ambiguous case exists, the students should be expected to provide both complete solutions to the triangle.

Examples/Activities

1. Solve each of the following triangles. State the given information and which parts of the triangle are yet to be solved.

   a) In △ ABC, a = 16 cm, b = 12 cm and c = 10 cm.

   b) In △ JKL, j = 9 cm, k = 8 cm, and m < L = 54°.

   c) In △ RST, m < R = 66°, m < S = 72°, and t = 6.4 cm.

   d) In △ WXY, m < W = 55°, m < X = 82°, and w = 14 cm.

Adaptations

Students can be given an assignment of these types of triangle solutions from many different resource texts. Each student might be given an assignment from a different text and asked to model one question from the text for the class.

This is an area where real-world problem situations may also be used. Students may have real-world examples that can be solved using these laws. This might be considered as a 'take-home' assignment. (IL)

1. List the given information from each triangle, and state whether it represents SSS, SAS, ASA, AAS, AAA, or SSA. Then completely solve each triangle.

   a) In △ ABC, m < A = 68°, m < C = 42°, and b = 16 cm.

   b) In △ DEF, d = 25 cm, e = 33 cm, and m < F = 41°.

   c) In △ GHJ, g = 18 cm, h = 8 cm, and m < G = 54°.

   d) In △ KLM, m < K = 72°, m < L = 57°, and m < M = 51°.

   e) In △ NPR, n = 18 cm, p = 27 cm, and r = 34 cm.

   f) In △ STV, m < S = 28°, t =29 cm, and s =14 cm.

   g) In △ WXZ, w = 38 cm, x = 14 cm, and z = 19 cm.

Note g) does not have a solution because the triangle as written does not exist. (14 cm + 19 cm < 38 cm)

Students can be given an assignment to work on at home and then asked to check each others' work in class the next day. If the students do not agree on a solution to any particular question, the solution can be done by the teacher as an example, or by some other class member.

Concept D: Applications of Trigonometry

Objectives/Skills

D.5
To solve word problems by means of the Law of Sines/Cosines.

Instructional Notes

Students should be given a set of problems/exercises that they can work at individually or in pairs. They can demonstrate correct solutions as examples, or raise difficult problems for discussion. Inasmuch as possible, these problems should be chosen for their applicability to real-world situations.

Students may also be instructed to design their own problem based on these laws and to submit it to others in the class for solution.

Examples/Activities

1. A surveyor wishes to determine the width of a pond. From a point A at one end of the pond he marks off a distance (B) of 150 m on one side of the pond, and a distance (C) of 167 m on the other side, so that B and C are directly across from each other. The angle at A between B and C is measured to be 43°. What is the distance across the pond?

2. A hockey net is 1.72 m wide. If a player stands 10 m directly in front of the left hand post, and he is to shoot at the net, what is the angle the player has to shoot at?

3. Two persons are standing 500 m apart on opposite sides of a building. One determines that the angle of elevation to the top of the building from their position is 22°, and the other determines it to be 27°. How tall is the building?

4. An emergency signal is being emitted from a fixed position. One response team determines the signal to be at a bearing of 63°, while a second response team, 2 km due east of the first team, determines the bearing to be 312°. Which team is closer to the emergency signal, and how far away is it?

5. Two streets meet at an angle of 74°. It is determined that no private lot should be allowed closer than 23 m from the intersection. To inhibit any high fences that would obscure vision, it is decided to plant a hedge about this triangular plot. What is the total length of the hedge on all three sides? What is the length of the longest side?

Adaptations

Students could be asked to research similar types of problems in various resource texts and to bring one or two of these to class for discussion, or to contribute to a class assignment.

Objectives/Skills

D.6
To determine the area of a triangle using $K = 1/2\ ab \sin C$,

$$K = \frac{a^2 \sin B \sin C}{2 \sin A}$$

or Heron's Formula,

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter of the triangle.

Instructional Notes

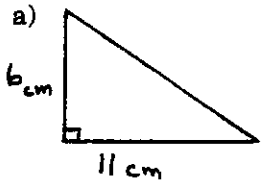
Students could be shown the derivation of the various area formulas (except Heron's) or they could be encouraged to attempt to develop the formulae on their own, individually or in small groups.

Once the students have encountered the formulae, they can attempt a series of exercises based on finding areas of triangles using these formulae.

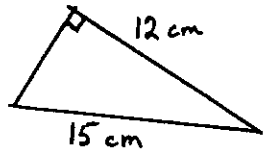
Examples/Activities

1. Determine the area of each of the following triangles:

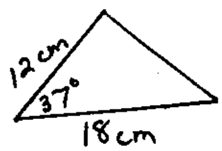
a)



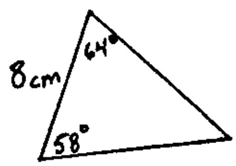
b)



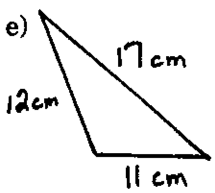
c)



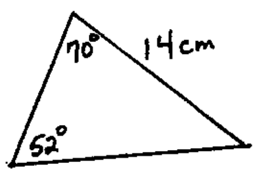
d)



e)



f)



2. Calculate the area of △ ABC, if
   a = 23 cm, b = 18 cm, and m < C = 48°.

3. Determine the area of △ DEF, if
   m < D = 54°, m < E = 63°, and f = 48 cm.

4. Calculate the area of an isosceles triangle whose base is 39 cm and whose base angles are 54°.

5. Determine the area of a triangle whose sides have lengths 25 cm, 33 cm, and 42 cm.

Adaptations

The proofs of the first two area formulas should be developed by the students, perhaps with some initial hints from the teacher. For advanced students, Heron's formula can be started and left for the students to carry out the algebraic manipulations required to complete the proof.

Objectives/Skills

D.7
To solve word problems involving objective 6.

Instructional Notes

Students can be given a set of problems to work at individually or in pairs. These problems can be found in most resource texts, and should be chosen to reflect real-world situations.

Once the exercise is complete, students can demonstrate a problem for the class, or identify a particular problem for class discussion.

Examples/Activities

Adaptations

1. A municipal crew is to seed a triangular plot to grass. The plot measures 60 m on one side, 73 m on a second side, and has an included angle of 59°. What is the area of the plot? If it is to be seeded at the rate of 80 grams of seed per square meter, how much seed is required?

2. A country has issued a stamp in the shape of an isosceles triangle, 25 mm on each of the two equal sides. If the vertex angle is 72°, what is the area of the stamp?

3. What amount of siding would be required to cover one end of an A-frame building whose base is 8 m and whose vertex angle is 58°?

Concept E: Trigonometric Identities

Foundational Objective:

- To demonstrate the ability to work with trigonometric identities and to be able to apply them when necessary (10 05 01). Supported by the following learning objectives.

Objectives/Skills	Instructional Notes
E.1 To prove and apply the reciprocal identities.	**Note:** While trigonometric identities are included in this course, it is not intended that they be taught in the traditional manner. Rather, it is recommended that the time line of two weeks be adhered to and that time be spent on applications of these identities to angle calculations in addition to verification. As well, verification exercises may be done as proofs, to reinforce the learning objectives of Concept A. The reciprocal identities are those whose product is 1. E.g.: $\cos\phi \sec\phi = 1$ or $\cos\phi = 1/\sec\phi$ $\sin\phi \csc\phi = 1$ or $\sin\phi = 1/\csc\phi$ $\tan\phi \cot\phi = 1$ or $\tan\phi = 1/\cot\phi$ These identities can be shown to be true by the students, using a unit circle (x,y,r) or by employing the right triangle approach. The students may be asked to use these identities to verify a statement, or to determine a specified value.
E.2 To prove and apply quotient identities.	The quotient identities are usually thought of as those identities where two or more functions are divided to obtain another function. For our purposes, these are; $\tan\phi = \dfrac{\sin\phi}{\cos\phi}$ and $\cot\phi = \dfrac{\cos\phi}{\sin\phi}$ Students can be asked to develop these quotient identities from the unit circle (x,y,r) or from right triangle trigonometry. They might work in small discussion groups to develop these. Once these are developed, the students can be given a set of exercises in which they are asked to verify statements or to determine specific values.

Examples/Activities

1. Show that each statement is true.

   a) $1/\sec \phi + 1/\cos \phi = \cos \phi + \sec \phi$

   b) $\csc \phi \cdot \sec \phi \cdot \sin \phi = 1/\cos \phi$

   c) $\dfrac{\cot \phi \cdot \tan \phi}{\csc \phi} = \sin \phi$

   d) $1/\csc \phi \cdot \cos \phi \cdot \sin \phi \cdot \sec \phi = \sin^2 \phi$

2. Determine the indicated value in each case.

   a) If $\cos \phi = \sqrt{3}/2$, then $\sec \phi =$?

   b) If $\cot \phi = 4.2$, then $\tan \phi =$?

   c) If $\csc \phi = -6.3$, then $\sin \phi =$?

3. Find all possible values of ϕ in each part of question 2.

Adaptations

Students may wish to show these statements true in ways other than step-by-step solutions. If some students wish to utilize tables, graphs, or other methods, they can be encouraged to do so. A discussion can be held about the number of methods possible. One outcome of such a discussion is determining if there is any one method more appropriate than others. (COM)

---

1. Verify each of the following statements.

   a) $\sin \phi \cot \phi = \cos \phi$

   b) $\tan \phi \csc\phi = \sec \phi$

   c) $\dfrac{\cos \phi - 1}{\sin \phi} = \cot \phi - \csc \phi$

   d) $\csc \phi (\cos \phi + \sin \phi) = 1 + \cot \phi$

2. Determine the value specified.

   a) If $\cos \phi = -3/5$ and $\sin \phi = 4/5$, what is $\tan \phi$? $\cot \phi$?

   b) If $\cot \phi = -3/2$, what are possible values for $\sin \phi$? $\cos \phi$? $\sec \phi$? $\csc \phi$?

3. Write the given function in terms of the specified function.

   a) $\csc \phi$; in terms of $\sin \phi$

Students may develop different processes for verifying these identities. Several of these might be displayed and discussed, in order that the class may see there are different approaches, and so that individuals may adopt a process that suits them.

Concept E: Trigonometric Identities

Objectives/Skills

Instructional Notes

E.3
To prove and apply the Pythagorean identities.

The Pythagorean identities can be developed from either the right triangle or the unit circle. The students could be given an opportunity to develop these on their own, or in small groups.

These identities are:

$\sin^2 \phi + \cos^2 \phi = 1$

$1 + \tan^2 \phi = \sec^2 \phi$

$1 + \cot^2 \phi = \csc^2 \phi$

Students should be asked to find alternative forms of each; such as,
$\sin^2 \phi = 1 - \cos^2 \phi$, in order to become familiar with some of the common variations of these identities.

Once students have developed these identities, a set of exercises on verification and determining specific values can be assigned.

E.4
To prove and apply the Addition/Subtraction identities.

The development of the addition/subtraction identities can be quite complex. It is suggested that the teacher model the development of the identity $\cos(ß - ∂) = \cos ß \cos ∂ + \sin ß \sin ∂$. This development can be found in a number of resource texts. It usually employs the distance formula.

When this identity has been developed, the others can be derived using this as a starting point. The other identities to be used are

$\cos(ß + ∂) = \cos ß \cos ∂ - \sin ß \sin ∂$

$\sin(ß + ∂) = \sin ß \cos ∂ + \cos ß \sin ∂$

$\sin(ß - ∂) = \sin ß \cos ∂ - \cos ß \sin ∂$

$$\tan(ß + ∂) = \frac{\tan ß + \tan ∂}{1 - \tan ß \tan ∂}$$

$$\tan(ß - ∂) = \frac{\tan ß - \tan ∂}{1 + \tan ß \tan ∂}$$

As an addition, the teacher may wish to use these formulas to show $\cos(-ß) = \cos ß$, $\tan(-ß) = -\tan ß$, and the like.

Examples/Activities	Adaptations

Examples/Activities

1. Verify each of the following statements.

   a) $\sin^2 \phi + \cos^2 \phi + \tan^2 \phi = \sec^2 \phi$
   b) $\dfrac{(1 - \cos^2 \phi)}{\sin \phi \cos \phi} = \tan \phi$
   c) $\cos^2 \phi - \sin^2 \phi = 2\cos^2 \phi - 1$
   d) $\dfrac{\csc^2 \phi - \cot^2 \phi + \tan^2 \phi}{\sec \phi} = 1/\cos \phi$

2. Express each of the following in simplest terms.

   a) $\sqrt{1 - \sin^2 \phi} =$
   b) $1 - (\sin^2 \phi + \cos^2 \phi) =$
   c) $\dfrac{(1 - \cos^2 \phi) \cot \phi}{\sin \phi} =$

3. Determine the indicated value.

   a) If $\tan \phi = 2/3$, what is $\sec \phi$?

4. Show the following identities are true by substituting exact values and simplifying.

   a) $\sin^2 30° + \cos^2 30° = 1$
   b) $\sec^2 45° - \tan^2 45° = 1$
   c) $1 + \cot^2 60° = \csc^2 60°$

Adaptations

Many of these types of exercises can be located in various resource texts. Students might be asked to research various texts and to propose exercises for solution by the class. (IL)

---

1. Use the addition/subtraction identities to obtain exact values for each of the following:

   a) $\sin 75°$
   b) $\cos 15°$
   c) $\tan 105°$

2. Simplify each, using the addition/subtraction identities.

   a) $\cos 153° \cos 63° + \sin 153° \sin 63°$
   b) $\sin 54° \cos 36° + \cos 54° \sin 36°$
   c) $\dfrac{\tan 110° - \tan 65°}{1 + \tan 110° \tan 65°}$

3. $< A$ and $< B$ are angles in standard position, such that $< A$ is in the fourth quadrant with $\cos A = \sqrt{5}/3$, and $< B$ is in the third quadrant with $\sin B = -\sqrt{3}/2$.
   Determine:
   a) $\sin (A + B)$
   b) $\tan (A - B)$

Students could be assigned some verification questions for intrinsic interest if they exhibit some skill in applying these identities. They could be instructed to attempt some of the questions found in various resource texts and to bring interesting or difficult exercises to the attention of the class. The class could then attempt a solution to the problem.

Concept E: Trigonometric Identities

Objectives/Skills	Instructional Notes

E.5
To prove and apply the Double-Angle identities.

The teacher can have the students derive the double-angle identities by substituting into the addition identities and simplifying.

The double-angle identities to be done are

$$\cos 2\phi = \cos^2 \phi - \sin^2 \phi, \text{ or}$$
$$= 1 - 2 \sin^2 \phi, \text{ or}$$
$$= 2 \cos^2 \phi - 1$$

$$\sin 2\phi = 2 \sin \phi \cos \phi$$

$$\tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi}$$

The students can begin with

$\sin (\phi + \beta) = \sin \phi \cos \beta + \cos \phi \sin \beta$, and substitute ϕ for β, which becomes

$$\sin (\phi + \phi) = \sin \phi \cos \phi + \cos \phi \sin \phi$$
$$\sin (2\phi) = 2 \sin \phi \cos \phi.$$

The other double-angle identities can be shown in a like manner.

Once the identities have been proved, a set of exercises on applications of these identities may be assigned.

E.6
To determine $\sin n\phi$, where n is a natural number.

Students can be instructed to develop expressions for $\sin 3\phi$, 4ϕ, 5ϕ, etc., based upon their knowledge of the identity for $\sin 2\phi$. They can do the same for $\cos n\phi$, and $\tan n\phi$, or simply conjecture what might happen. They should test their conjecture.

The above will probably not have much meaning in itself, except for some intrinsic interest. Therefore, students should also attempt to graph $\sin \phi$, $\sin 2\phi$, $\sin 3\phi$, $\sin 4\phi$, etc., for values of ϕ from 0° to 360° and compare the graphs.

The students may also explore the relationships of $\sin n\phi$ by preparing charts (tables of values) for ϕ from 0° to 360° and observing the results.

The teacher may assign various groups a different method of exploring $\sin n\phi$ and have the groups compare their results. The presence of more than one depiction should help students understand the effect of a multiplier on ϕ. These can also be done for $\cos n\phi$, $\tan n\phi$, time permitting, or as a project outside class time.

Examples/Activities

1. Determine the value of each expression.

   a) $2 \sin 22.5° \cos 22.5°$

   b) $\cos^2 67.5° - \sin^2 67.5°$

   c) $\dfrac{2 \tan 75°}{1 - \tan^2 75°}$

2. If $\sin \phi = 3/5$ and ϕ terminates in the second quadrant, determine the values of:

   a) $\sin 2\phi$

   b) $\cos 2\phi$

   c) $\tan 2\phi$

---

1. Find an expression for $\sin 4\phi$.

2. Develop an expression for $\cos 3\phi$.

3. Write an expression for $\tan 3\phi$.

4. Draw the graphs of $y = \sin \phi$, $y = \sin 2\phi$, and $y = \sin 3\phi$ on the same coordinate axes. What is the wavelength of each?

5. Draw the graphs of $\tan 2\phi$ and $\tan \phi$ on the same axes. Note your observations regarding the similarities and differences of the two graphs.

6. Draw the graphs of $\cos \phi$ and $\cos 5\phi$ on the same axes. What similarities and what differences are there?

7. If you were given $\sin n\phi$, how would you describe the effect of n on the graph of the sine function? What might happen if n were a rational number between 0 and 1?

Adaptations

In addition to applying the double-angle identities, some teachers may wish, time permitting, to have students do a few verifications involving double-angle identities. These can be found in resource texts and might be assigned to students to work on in small groups.

Concept E: Trigonometric Identities

Objectives/Skills

E.7
To apply the Half-Angle identities.

Instructional Notes

In this section, students can be given the half-angle identities, and asked to use them in applications. These half-angle identities are:

$$\left|\cos\frac{\phi}{2}\right| = \sqrt{\frac{1+\cos\phi}{2}}$$

$$\left|\sin\frac{\phi}{2}\right| = \sqrt{\frac{1-\cos\phi}{2}}$$

$$\left|\tan\frac{\phi}{2}\right| = \sqrt{\frac{1-\cos\phi}{1+\cos\phi}}$$

Examples/Activities

1. Use the half-angle identities to determine the exact value of:

   a) sin 15°

   b) cos 15°

   c) tan 15°

   d) sin 22.5°

   e) cos 7.5°

2. Given that tan $\phi = 4/3$ in the third quadrant, determine the value of sin $\phi/2$, cos $\phi/2$, and tan $\phi/2$.

3. Given that sin $\phi = 5/13$ in quadrant I, find the values of sin $\phi/2$, cos $\phi/2$, and tan $\phi/2$.

Adaptations

If time permits, and if students exhibit some proficiency with identities, some verification exercises could be done with half-angle identities.

Concept F: Trigonometric Equations

Foundational Objectives:

- To demonstrate understanding and ability in solving trigonometric equations (10 06 01). Supported by the following learning objective.

Objectives/Skills

F.1
To solve trigonometric equations by finding a particular solution and by finding the general solution.

Instructional Notes

Equations requiring a particular solution have a restricted domain and the solution(s) obtained are required to be within that domain. General solutions are expected where no restriction on the domain is given. Hence, values which repeat over the extended domain must be included. The students must discern the pattern of repetition and account for this repetition in their answer. The general solution, therefore, usually requires a variable multiplier in its answer.

It is recommended that the teacher begin with some simple equations involving one trigonometric function, and one value, such as $\sin \phi = 1/2$. The particular solution for $0° \leq \phi < 360°$ can be done first and the general solution shown as well.

It is also recommended that the students be instructed to draw the graph of $\sin \phi$, and determine values of ϕ when $\sin \phi = 1/2$. Various groups in the class can be asked to attempt different methods of solution to the same question and to compare results among groups. Graphing, algebra, tables, and circle graphs are some methods that might be employed.

Examples/Activities

1. Solve each of the following, first for the particular solution where $0° \le \phi < 360°$ and then for the general solution.

   a) $\sin \phi = \sqrt{3}/2$

   b) $\tan \phi = 4.2$

   c) $\sin 2\phi = -\sqrt{3}/2$

   d) $\cos 3\phi = 1$

   e) $\cos \phi/2 = 1/2$

   f) $\tan 2\phi = \sqrt{3}$

   g) $\sin^2 \phi - 1/2 \sin \phi = 0$

   h) $2 \tan^2 \phi = 1 - \tan \phi$

   i) $\cos^2\phi - 3 \cos \phi + 2 = 0$

   j) $\sin 2x - \cos x = 0$

Adaptations

Students can be instructed to solve equations showing at least two different methods of solution and to complete a hand-in assignment in this manner. They may be asked to attempt additional questions from other resource texts and to share interesting and/or difficult exercises with the remainder of the class.

These exercises can all be adapted by asking students what effect a phase shift would have on the results of a sample question. What is the effect of different restrictions on the domain?

Appendices

Appendix A

Instructional Strategies and Methods

The following précis describes methods from *Instructional Approaches: A Framework for Professional Practice* (1991). Refer to that document for a more detailed discussion of these methods and the "families" of strategies to which they belong.

Cooperative Learning

This is an approach where students work together to complete a task or project which is often based on their particular group's strengths and interests. Students engage in brainstorming, reflective discussion, mutual decision making, or conducting research. The purpose of using cooperative learning groups is to minimize competitiveness and feelings of low self-esteem and to increase students' respect for and understanding of each others' unique abilities, interests, and needs. (PSVS)

Concept Attainment

Concept attainment focuses on understanding what characteristics or attributes may be useful for distinguishing between members and nonmembers of a group or class. Clues are supplied by the teacher from which students are to determine the identity of concepts. The five key elements necessary to define a concept are: names, examples, attributes, attribute values, and rules. (CCT)

Lecture

The learning environment during lecture is task centred, teacher directed, and highly structured. In general, this strategy involves the teacher explaining a new concept to the large group, testing student-understanding by controlled practice, continued guided practice under the teacher's supervision, and finally independent practice by the student.

Problem Solving

Problem solving is the process of accepting a challenge and striving to resolve it. It allows students to become skillful in selecting and identifying relevant conditions and concepts, searching for appropriate generalizations, formulating plans, and employing acquired skills. The process of problem solving involves three phases: understanding the problem, devising and carrying out the plan, and looking back. (CCT)

Learning Centres

Learning centres may include examples of print and non-print materials, types of audio-visual equipment, and programmed instruction. The purpose is to provide students with differentiated learning experiences in the form of individual or group activities. Students may be directed by the teacher or may be given the opportunity to select, manage, and evaluate the experiences around which the centre was designed. (IL)

Drill and Practice

Drill and practice can be of many types; visual, manipulative, oral, written, or any combination of these. To be of most value it must always be accompanied with good mental processes. Drill and practice should follow the developmental and discovery stages of learning and be used to reinforce and extend basic learning.

Compare and Contrast

This strategy develops the students' ability to collect, organize, and remember information and helps them apply that information for new learning. This method consists of three phases: the description phase, comparison phase, and application phase. By keeping the students actively and thoughtfully involved in collecting and processing information, the compare and contrast method helps them to develop the ability to become independent learners. (CCT)

Role Playing

Role playing allows students to deal with problems through action: through identifying the problem, acting it out, and discussing it. The essence of role playing is the involvement of participants and observers in a real problem situation and the desire for resolution and understanding that this involvement generates.

Games

A well-designed game will partly teach itself. Simulations enable students to learn first-hand from the simulated experiences built into the game rather than from teacher's explanations. However, it is still important that a teacher raise the students' consciousness about the concepts and principles involved in the game, because the students may not always be aware of what they are learning and experiencing.

Projects

Projects are used for independent study and should supplement or enrich the basic lesson, once the skills or concepts have been learned. They foster the development of individual student initiative, self-reliance, and self-improvement. The role of the teacher is one of resource person rather than a presenter. (IL)

Computer-Assisted Instruction

This independent study method promotes learning through interactive demonstration, drill and practice, tutorial, simulation, educational games, and programming as problem solving. Teachers must carefully assess the software under consideration so that it meets the desired educational goals.

Tutorial Groups

Tutorial groups are set up to help students who need remediation or additional practice, or for students who can benefit from enrichment. A tutorial group is usually led by the teacher. Tutorial groups provide for greater attention to individual needs and allow students to participate more actively. Peer tutoring occurs when a student (the tutor) is assigned to help other students (the learners). The roles played by teacher, tutor, and learner must be explained and expectations for behaviour must be outlined.

Appendix B

Mathematics A 30

Concept B: Relations and Functions

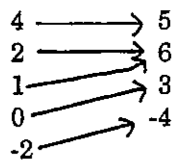
Objectives	Instructional Notes
B.1, B.2, B.3 (Mathematics 10) To display a relation as a set of ordered pairs, mapping diagram, and a graph.	Students should obtain some practice in this concept by being given a relation and being asked to provide ordered pairs which represent the relation. They might also be given a restricted domain, that would limit the number of ordered pairs. Given a relation in one of the three forms listed in this concept, the students should be expected to provide the other two. Initially, they might work together, but could be expected to provide these individually after some practice. They might be given a series of real-world situations to demonstrate in these three ways and be asked to conclude which of the three methods is most useful to them in each situation.

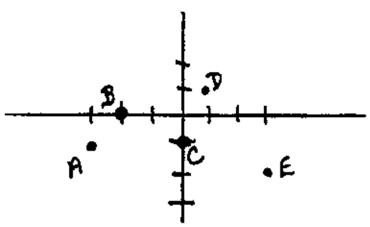
Examples/Activities

Express each of the following as a set of ordered pairs.

1. $\{(x,y):\ y = 2x-3,\ x \in \{0,1,2,3,4\}\}$

2. $\{(x,y):\ y = 3x+4,\ x \in I, x>2\}$

3. 

4. 

Adaptations

There are many opportunities for students to extend this topic. Teachers may require students to find examples of relations in the real-world through newspapers, magazine articles, or books, and to illustrate them for the class.

Examples can be drawn from many walks of life, politics, economics, sociology, sports, business, sales, statistics, health, agriculture, and from any sector of geography, right from individual concerns to global scenarios.

Students might be expected to give written background information on their examples and thus utilize writing skills in mathematics.

Express each of the above (except 3) as a mapping diagram.
Express each of the above (except 4) as a graph.

Express each of the following as ordered pairs, a mapping diagram, and as a graph. Which is(are) the most useful descriptions for each case?

1. Scores in a recent tennis match were 7-5, 3-6, 4-6, 6-2 and 6-4 .
2. The approval ratings for a politician in a series of polls were as follows:
   Jan For 58% Vs. 42%
   Mar For 53% Vs. 47%
   May For 51% Vs. 49%
   July For 46% Vs. 54%
   Sept For 48% Vs. 52%
   Nov For 43% Vs. 57%
3. When rolling a pair of dice thirty-six times, the following results were obtained: (no. of rolls, total),
   { (1,2),(2,3),(3,4),(4,5),(5,6),
   (6,7),(5,8),(4,9)(3,10),(2,11),(1,12) }.

Concept B: Relations and Functions

Objectives	Instructional Notes
B.4 (Mathematics 10) To graph relations representing real-world situations.	Many examples of these types of relations can be found in most texts, but students may prefer to obtain and use examples that affect them more closely, or are of interest to them. They can be instructed to research these from other sources, or to expand upon those they found in previous objectives. Students working cooperatively could screen suitable examples, and prepare graphs for presentation to the entire class. Discussion could ensue on any of the topics that might elicit interest, and be of use to students. (PSVS) Students may also use this concept to explore individual situations that lend themselves to this topic. (E.g.: number of shots, number of baskets)
B.1, B.9 (Mathematics 10) To determine the domain and range of a relation.	This concept should enable students to determine the domain and range of a relation in several different contexts. Exercises can focus on mathematical solutions, visual solutions, extrapolation, or sorting out information. Students could be presented with a variety of exercises and be asked to discuss in small groups the meanings of domain and range. Using these meanings, they could identify the domain and range in the various contexts stated above. (COM) Each group member should be satisfied with the results obtained by the group and should be able to explain the procedure used in each situation. The class as a whole may summarize the results, and a series of exercises assigned for practice. This topic is especially valuable for students, as proper identification of domain and range lends itself to many topics in analytical mathematics and in curve sketching in calculus.

Examples/Activities

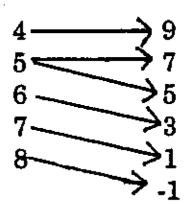
Graph each of the following. What observations might you make for each case?

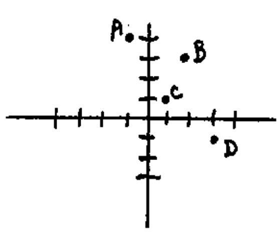
1. The number of wins and losses by baseball teams in the American League East at the most recent report. (If not in season, use any other athletics statistics.)
2. The numbers of cars and trucks sold by the top ten sales agents at a local car lot.
3. The amount of gasoline used by the same vehicle (in litres/100 km) at different speeds.
4. The cost of resurfacing streets of varying lengths at the present cost per kilometre.
5. The number of students in your class/school who get marks in the 50s, 60s, 70s, 80s, 90s.
6. The number of students who participate in a school's activities, and the cost of running each activity.

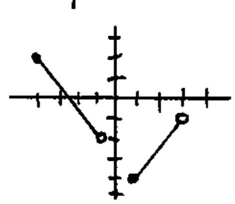
Adaptations

Students may wish to focus on some of their examples, and discuss the results. This might lead to identification of some improvements, or alternatives, in dealing with certain situations. If this occurs, it may present an opportunity to combine the mathematics with writing and real-world problem-solving skills.

How would the domain and range be determined in each of the following situations? What is the domain and range for each?

1. Given $\{(x,y): y = 3x-5,\ x? \in (-2, -1, 0, 1)\}$
2. $\{(-3,2), (-4,5), (-2,0), (0,2), (1,3), (2,5), (0,3)\}$
3. $\{(x,y): y = -4x+3, x< 5\}$
4. 

5. 

6. 

Students could refer to questions done in previewing objectives, to determine domain and range of each.

A discussion regarding the possible value of domain and range in various situations might be enlightening to some students.

Concept B: Relations and Functions

Objectives	Instructional Notes
B.7, B.8 (Mathematics 10) To determine if a set of ordered pairs represents a function.	The definition of function was introduced in Mathematics 10. Students should review this definition and obtain some practice in determining whether a set of ordered pairs represents a function by definition. Exercises given to students should include some non-functions as well as functions. Students should be expected to not only identify these sets as representing functions or not, but should also be able to demonstrate why they are, or are not, functions.
B.8 (Mathematics 10) To determine if a relation is a function by applying the vertical line test.	Give students the graphs of some functions and some non-functions to examine (including some where both the domain and range are integers). Have students list a set of ordered pairs that belong to each graph. Then have them determine, with justification, whether the graph represents a function. Have the students separate the graphs of functions from those that are non-functions. This can be done in small groups or on the board as an entire class. Discuss any characteristics each set of graphs have that might seem to make it distinct from the other set. The students should reach the conclusion that if any vertical line superimposed on a graph intersects the graph in more than one point, it is not the graph of a function. This is the vertical line test. If students are not able to reach this conclusion, the teacher should provide more graphs, hints, and directed questions to lead the students to this conclusion.

Examples/Activities

Adaptations

Determine whether each of the following represents a function. Be prepared to justify your answer.

1. {(1,3),(2,5),(3,7),(4,9),(5,7)}

2. {(-2,0),(-1,-1),(0,-2),(1,-3),(2,5)}

3. {(2,-4),(3,-4),(4,-4),(5,-4),(6,-4)}

4. {(2,3),(2,1),(2,-1),(2,-3)}

5. {(2,7),(3,6),(4,5),(5,4),(6,3),(4,7),(7,4)}

6. {(3,5),(4,4),(5,3),(6,2),(5,3),(7,1)}

For an exercise, draw several graphs on the board (overhead, etc.) and ask students to determine whether they represent functions. Ask them to justify their response.

Students in small groups could use different resource texts and be asked to locate a specific number of graphs that represent functions and a specific number that did not. These examples should be recorded by page and number and should be justified to each group member. (IL)

Provide students with a set of ordered pairs, instruct them to plot these ordered pairs, and then apply the vertical line test to determine if the ordered pairs represent a function.

Concept B: Relations and Functions

Objectives	Instructional Notes
B.13 (Mathematics 10) To graph linear equations using the x- and y-intercepts.	Present students with a linear equation and ask them what type of geometric figure it would represent. Most should recognize this equation as representing a line. Ask them how many points it takes to determine a line. Then ask them to determine two points on the line. Using some group's example, plot these two points and draw a line through them. Ask students to identify the coordinates of the points where the line intersects the two axes. Define these as the x- and y-intercepts. Give students two or three other equations and ask them to determine the intercepts, plot them, and draw the line joining them. Get solutions from various groups. Once students have developed the ability to graph using the intercepts, a short exercise can be assigned.
B.14 (Mathematics 10) To graph linear equations using one intercept and the slope.	Have students graph several equations: by plotting, using intercepts, graphic calculators, or computers. Have them make note of the intercepts and the slope. Provide students with one intercept and the slope for a few exercises. Ask them to draw the graph with this information. They can check their graphs with other members of their group. Once students can do these in small groups, a few exercises can be given as an assignment.

Examples/Activities

Graph each of the following using the x- and y-intercepts.

1. y = 3x-4

2. 2x+5y = 10

3. 3x-4y = -12

4. x = 3y-2

5. (1/2) x+ (3/4) y = 6

6. For lines such as x = 5, y = -3, and 4x-3y = 0, have students identify the intercepts, and discuss how to recognize situations where other points are necessary to plot the line.

The examples and exercises could be similar to the following.

1. Draw the line as described in each of the following;

   a) slope(m) = 3, y-intercept(b) = 4

   b) m = 1/2, b = -3

   c) m =-2, x-intercept(a) = 3

   d) m = 3/5, a =-1

   e) m = 0, b = 3

   f) m = undefined, a = -4

2. Draw the graph of each of the following, using the slope and one intercept.

   a) y = 2x-3

   b) y = (1/4)x +5

   c) 2x-3y = -6

   d) 4x+3y = 0

   e) (2/3)x -(1/2)y = 4

3. On a coordinate map, a plane's navigator finds that on a line due north, at the town of Coon Rapids, the navigator must turn 30° to the East (slope of 2) to follow her projected route. Have students depict this situation on a graph.

Adaptations

Students may also wish to discuss the advisability of checking their calculations by using a third point in each case. Are there other checks they can utilize?

Student groups can be given paper, pencils, rulers, and protractors and asked to use their new-found skills to address situations such as: how can a person mark out a baseball diamond, if he/she knows the location of home plate and the pitcher's mound (66 feet apart), that the third base line (m=-1) and first base line (m=1) intersect at home plate, and that third base and first base are each 90 feet from the home plate? This can be extended by having students complete the baseball diamond by including the outfield. What if it had to be superimposed on a football field? What are some restrictions that might apply?

Appendix C

Permutations and Combinations

When introducing this topic, use nCr or

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{n!}{r!(n-r)!}$$

but

$$\begin{bmatrix} n \\ r \end{bmatrix}$$

certainly should be used somewhat as it is almost a universal symbol for combinations and is particularly used when discussing the binomial theorem.

First a lottery problem for them - for p12?

In Lotto 6/49 what is the probability of
a) picking the winning ticket?
b) picking none of the right numbers for your ticket?
c) getting exactly one of the six numbers correct?

a)

$$p = 1/\begin{bmatrix} 49 \\ 6 \end{bmatrix} = 1/13,983,816$$

b)

$$p = \frac{\begin{bmatrix} 6 \\ 0 \end{bmatrix}\begin{bmatrix} 43 \\ 6 \end{bmatrix}}{\begin{bmatrix} 49 \\ 6 \end{bmatrix}} = \frac{6,096,454}{13,983,816} = 4360$$

c)

$$p = \frac{\begin{bmatrix} 6 \\ 1 \end{bmatrix}\begin{bmatrix} 43 \\ 5 \end{bmatrix}}{\begin{bmatrix} 49 \\ 6 \end{bmatrix}} = \frac{6(962,598)}{13,983,816} = 4130$$

Note that the chances of at most 1 correct is seen to be about .849. I think you win $10 for three correct. What are the chances in this case? I think p is about .0177, roughly a one in 57 chance.

To illustrate the size of the numbers involved you might suggest the following - suppose you always buy the same six numbers on Wednesday and Saturday, roughly 100 tickets per year. On the average that particular combination will come up once in about (1/2)(14,000,000)/100 = 7000 years!

Appendix D

Binomial Theorem

The Binomial Theorem for positive integer exponents should really be couched in terms of combinations. Consider $(x + y)^4$. If we think of this as long multiplication, we first observe that from the product $(x + y)$ $(x + y)$ $(x + y)$ we will get terms such as x^4, x^3y, x^2y^2, xy^3, and y^4. How do we get x^4? By choosing 4 x's out of 4 x's in the x + y terms; e.g.: in

$$\begin{bmatrix}4\\4\end{bmatrix} = 1$$

ways; x^3y? By choosing 3 x's out of 4 x's in the x + y terms; e.g.: in

$$\begin{bmatrix}4\\3\end{bmatrix} = 4$$

ways; x^2y^2? by choosing 2x's our of 4x's in the x + y terms; e.g.: in

$$\begin{bmatrix}4\\2\end{bmatrix} = 6$$

ways; xy^3?

By choosing 1 x out of 4 x's in the x + y terms; e.g.: in

$$\begin{bmatrix}4\\1\end{bmatrix} = 6$$

ways; y^4? By choosing 0 x's out of 4x's in the x + y terms; e.g.: in

$$\begin{bmatrix}4\\0\end{bmatrix} = 1$$

ways. In summary we have

$$(x+y)^4 = \begin{bmatrix}4\\4\end{bmatrix}x^4 + \begin{bmatrix}4\\3\end{bmatrix}x^3y + \begin{bmatrix}4\\2\end{bmatrix}x^2y^2 + \begin{bmatrix}4\\1\end{bmatrix}x\,y^3 + \begin{bmatrix}4\\0\end{bmatrix}y^4 = x^4 + 4x^3y + 6x^2y^2 + 4x\,y^3 + y^4$$

In general to find the coefficient of x^ky^{n-k} in $(x + y)^n$ it is simply the number of ways of choosing k x's from nx's. E.g.:

$$\begin{bmatrix}n\\k\end{bmatrix}$$

Introduce the Binomial Theorem this way and then have students do $(x + y)^1$, $(x + y)^2$, (x + y)3, $(x + y)^4$, $(x + y)^5$, $(x + y)^6$ etc. to see the pattern of coefficients. Show them Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1 etc.

Maybe some good students will then deduce Pascal's Identity upon which the triangle is based; namely,

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r \end{bmatrix} + \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}$$

Ask them to sum the rows of the triangle. Why is it always 2^n? This really relates to sets and subsets that they have seen earlier. For example, if we have four people, in how many ways can we pick people to make a committee of size 0, of size 1, of size 2, of size 3, of size 4? These are

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4, \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 6, \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 1$$

respectively. Together they add up to 16. Equivalently, how many subsets (including the empty set) does a set of size four have? $16 = 2^4$. Why do we get 2^n if we replace four by n? It is because the Binomial Theorem with x = y = 1 tells us that the total number is

$$2^n = (1+1)^n = \begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} ... + \begin{bmatrix} n \\ n-1 \end{bmatrix} + \begin{bmatrix} n \\ n \end{bmatrix}$$

Here is a similar challenge. Instead of adding across a row of the triangle, alternately insert + and - signs. E.g.: in row four you get 1 - 4 + 6 - 4 + 1 = 0. Why do we always get 0? It is because the Binomial Theorem with x = 1 and y = -1 tells us that the total is

$$0 = (1-1)^n = \begin{bmatrix} n \\ 0 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} - ... + (-1)^{n-1} \begin{bmatrix} n \\ n-1 \end{bmatrix} + (-1)^n \begin{bmatrix} n \\ n \end{bmatrix}$$

Note this illustrates why the number of subsets each with an even number of elements equals the number of subsets each with an odd number of elements. E.g.:

$$\begin{bmatrix} n \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} + \begin{bmatrix} n \\ 4 \end{bmatrix} + ... = \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 3 \end{bmatrix} + \begin{bmatrix} n \\ 5 \end{bmatrix}$$

Appendix G

Mathematical Proof - Introduction and Student Notes

This resource package contains lessons to help the teacher with the unit on proofs in Mathematics C 30. For students to be able to do each question in this unit, it would be desirable for them to take this as their final unit. However more than 95% of the questions contained here can be done without the knowledge of the "trigonometry" portion of Mathematics C 30. Hence, teachers may use this as a first unit in the course as well.

The resource consists of three sections -- Appendix G (Student Notes), Appendix H (Teacher Notes), and Appendix I (Assignment Solutions).

Appendix G is intended to be copied and given to the students. This section contains lesson notes with the **examples not worked out**. The idea is that the teacher would discuss the material and work through the examples and conduct the suggested activities. The students would fill in the missing pieces as the discussion ensues.

Appendix H and I are for use by the teacher. Appendix H indicates, with the **examples fully worked**, what the author had in mind when creating the student notes. Appendix I contains the **full solutions to the assignments**. Teachers can withold these solutions from students or photocopy them as desired.

Because proofs of identities occurs in the trigonometry section, no additional identity proofs have been developed in this package. It is suggested that the package be done in the following order:

Lessons
1. The Need For Proof
2. Inductive Reasoning
3. Deductive Reasoning
4. Conditional Statements and Proofs By Counterexample
5. Integer Property Proofs
6. Deductive Geometric Proofs and Indirect Proofs
7. Coordinate Geometry Proofs
8. Proofs By Mathematical Induction

The amount of time available will dictate how many questions you will assign. In some cases, just working through the notes with the students will give them the idea of the concept under discussion and perhaps only a few practice questions need be given.

Every effort has been made to be accurate. The exercises included in this package have been used in the classroom with success while teaching Mathematics C 30. These materials will give students a better understanding of the topic. Each lesson may take several periods to complete.

Examples/Activities

Adaptations

Determine whether each set of data represents a hyperbola, an ellipse, or other. Then write the equation for any set that represents a hyperbola or an ellipse.

1. Foci $(\pm12,0)$; sum of focal radii = 26
2. Foci $(\pm5,0)$; difference, focal radii = 8
3. Vertices $(0,\pm7)$; Foci $(0,\pm5)$
4. Foci on x-axis, $a = 6$, $b = 4$
5. Foci on y-axis, $a = 5$, $b = 4$
6. Focus $(3,-2)$; directrix $x = -3$
7. Ellipse with centre at $(0,0)$, a domain extent of ±5 and range extent of ±3.
8. Ellipse with foci at $(3,2)$ and $(-5,2)$, and a constant sum of 10

1. If the general equation of an ellipse is given by $16x^2 + 25y^2 - 64x + 150y = 111$, and one of the foci is known to be $(5,-3)$; where is the other focus?

As an alternative, students could be asked to locate these types of problems in other resource books, and pose them to the class. A short discussion of suitability of such problems identified, and a class solution for each, might be useful. (COM)

Concept B: Conic Sections

Objectives/Skills	Instructional Notes
6.a) To determine the equation of a hyperbola or an ellipse, given sufficient information.	The students can be given a set of exercises in which they must examine the given information to determine whether it represents a hyperbola or ellipse. They should write the correct equation for this information. They might compare their results to others in the class, or work in pairs to decide which conic it represents.
6.b) To solve word problems involving ellipses or hyperbolas.	Students should be able to work with any problems in their text dealing with conic sections. Each text should have one or two problems relating to these conics. These may be assigned or done together as a class.

Examples/Activities

Determine the equation of the hyperbola in intercept form in each of the following cases. Also determine the equations of the asymptotes for each.

1. Foci $(\pm4,0)$; $2a = 6$
2. Foci $(0,\pm3)$; $2a = 2$
3. Foci $(3,0)$ $(11,0)$; $2a = 4$
4. Foci $(0,-3)$ $(0, 9)$; $2a = 10$
5. Foci $(4,-2)$ $(-2,-2)$; $2a = 4$
6. Foci $(-3,-7)$ $(-3,3)$; $2a = 6$

Rewrite the equation of each hyperbola in intercept form. Determine the asymptotes for each and sketch each hyperbola. Identify the vertices, centre, domain, and range of each hyperbola.

1. $2x^2 - 3y^2 = 6$
2. $x^2 - y^2 = 6$
3. $25x^2 - 4y^2 = 100$
4. $4x^2 - 2y^2 = -8$
5. $4x^2 - 9y^2 + 16x + 18y = 29$*
6. $9y^2 - 4x^2 + 18y + 24x = 63$*
7. $4x^2 - y^2 - 8x + 6y = 9$*

* Graphing can be done using the translation of axes.

Adaptations

Students who might be interested in a challenge could be given the task of graphing and analyzing hyperbolas whose general equation includes an xy term. They might employ technology to assist in the graphing, and then be asked to draw conclusions about the effect that the xy term has on the graph. (CCT) (TL)

A starting point in this case might be an equation of the type $xy = 12$.

Concept B: Conic Sections

Objectives/Skills

5.b)
To determine the equation of a hyperbola in the intercept form when given the foci and the difference of the focal radii.

Instructional Notes

The intercept form of the equation of a hyperbola is $x^2/a^2 - y^2/ b^2 = 1$ when the transverse axis is horizontal, and $y^2/a^2 - x^2/b^2 = 1$ when the transverse axis is vertical. The relationship of a, b, and c for the hyperbola is different than for the ellipse. Students should be shown this. For the hyperbola, the distance from the centre to the focus is c, the distance from the centre to the vertex is a, and $b^2 = c^2 - a^2$.

Students should also be shown how to determine the equations of the asymptotes given the intercept form of the equation of the hyperbola.

5.c)
To convert a given general equation for a hyperbola to the intercept form and sketch and analyze its graph.

In this section, the students should have practice in converting the general equation to intercept form, sketching the asymptotes, and sketching the graph of the hyperbola. Converting the equation to intercept form should be a relatively routine matter of employing the process of completing the square.

The sketching of the asymptotes could be done as a review of slope and lines. The determination of the equations of the asymptotes should be emphasized. Once the asymptotes are sketched, the graph of the hyperbola can be sketched as well. Students can compare their graphs with others, or with graphs obtained through various technologies, to observe their progress.

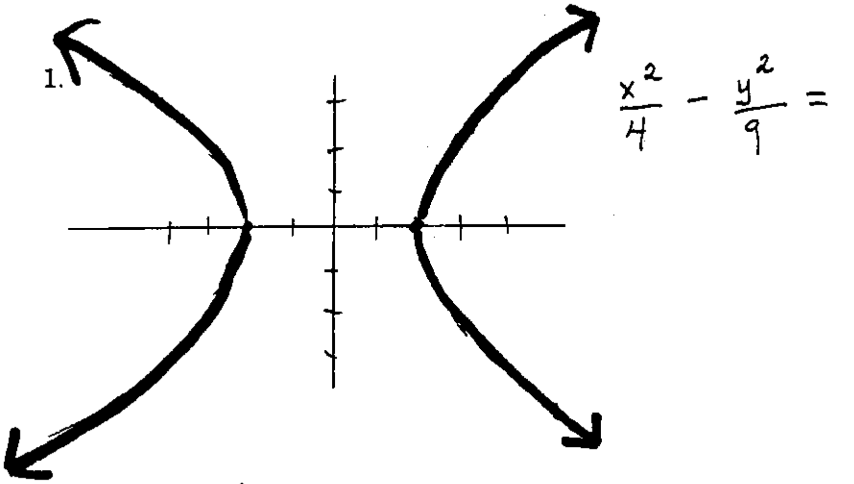
An analysis of the hyperbola might include the vertices, the centre, the asymptotes, domain, and range.

Examples/Activities

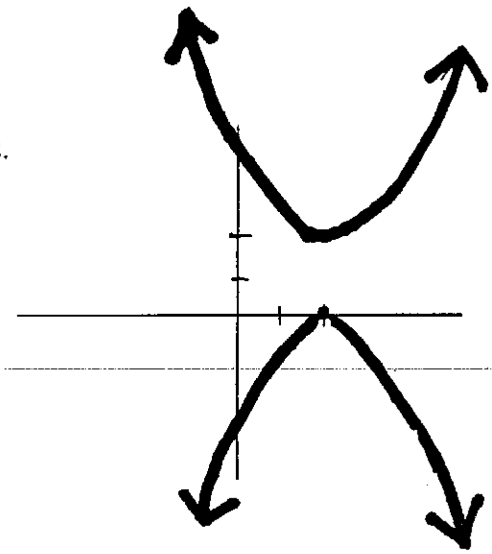
Adaptations

In the graphs provided, label the foci, centre, vertices, and draw in the asymptotes in each case.

1.



2.



3. Given that the foci of a hyperbola are (-3,0) and (3,0), and that the vertices are (-2,0) and (2,0), sketch the hyperbola. Label the centre and the asymptotes.

Objectives/Skills

5.
Hyperbola

a) To define and illustrate the following terms: hyperbola, foci, focal radii, major axis, minor axis, axis of symmetry, asymptotes, vertices, rectangular hyperbola.

Instructional Notes

The teacher might introduce hyperbolas by having the students draw the locus of a point whose given distance from two fixed points is a constant difference. Note that students may have to be given the fact that there are two parts to this locus.

Sketches of several of these loci could be drawn on the board or overhead and used to illustrate the main definitions associated with hyperbolas. The definition of transverse axis (the line segment connecting the two vertices) might also be useful.

Students should obtain some practice with these definitions and in identifying them from sketches of hyperbolas. Asymptotes of the hyperbola should be sketched. The equations of the asymptotes should be found in each case.

Examples/Activities

Convert each of the following to the intercept form of an ellipse, analyze the resulting equation and sketch the graph of the intercept form of the equation.

1. $9x^2 + 25y^2 = 225$

2. $16x^2 + 41y^2 - 246y - 297 = 0$

3. $9x^2 + 25y^2 - 36x - 50y - 164 = 0$

4. The orbit of the Earth is somewhat elliptical. If we placed the Earth and the sun on a grid such that the centre was at the origin, the sun was at one foci (1.2×10^6 km,0), and if the distance from the Earth to the sun (1.5×10^6 km) is the focal length, what is the equation of the Earth's orbit? [Most orbits are elliptical, and satellites, the moon, other planets, etc. can be used in similar problems. Students may have to research some of the distances.] An alternative is to write a sci-fi story regarding the second focus of the elliptical orbit of the Earth -- a black hole, a dark twin star, etc.

Adaptations

The teacher may adapt this section by introducing a coefficient for the xy term in $Ax^2 + By^2 + Cx + Dy + Exy + F = 0$. The effect of this term might be studied on the graph, the equation, the analysis, or other.

The teacher may also wish to have students note the coefficients of the x^2 and the y^2 terms throughout this unit, to determine what effect, if any, they have on the shape of the graph.

The notion of eccentricity can also be used in the study of the ellipse.

Concept B: Conic Sections

Objectives/Skills

4.c)
To convert a given general equation for an ellipse to the intercept form and sketch and analyze its graph.

Instructional Notes

The teacher may wish to have students first work on a few examples whose centres are the origin, to get some practice in converting to the intercept form with questions where the pattern is easy to establish. The analysis could include the foci, major and minor axes, domain, and range of the ellipse. A sketch should also be drawn with each question.

Once the students understand the exercise, the teacher may introduce exercises whose centre is not on the origin, but where one coordinate of the centre is zero. This will allow the students to obtain further practice in completing the square and in internalizing the steps needed in this section.

The teacher should be able to introduce general equations where the centre is (h,k). The students should be expected to convert to standard form, analyze, and sketch these graphs.

Examples/Activities

Adaptations

Write the equation of the ellipse given the information in each case.

1. Foci (4,0) (-4,0); sum, focal radii = 10
2. Foci (5,0) (-5,0); sum, focal radii = 26
3. Foci $(0,\pm4)$; sum, focal radii = 16
4. Foci $(0,\pm2)$; sum, focal radii = 6
5. Foci $(\pm3,0)$; 2a = 12
6. Foci $(0,\pm5)$; 2a = 14

Write the equation for each of the following ellipses.

1. Foci at (1,5) and (5,5); 2a = 6
2. Foci at (3,-2) and (3, 4); 2a = 10
3. Centre (4,-1); 2a = 8, b =1
4. Centre (3,5); 2a = 14, c = 5

For 3 and 4, assume the major axes are horizontal. If the teacher wishes, these may be done without this assumption. The students are expected to note that there are two possibilities.

Concept B: Conic Sections

Objectives/Skills

4.b)
To determine the equation of an ellipse in the intercept form when given the foci and the sum of the focal radii.
$x^2/a^2 + y^2/b^2 = 1$

Instructional Notes

The teacher may introduce this section by using ellipses centred on the origin and whose axes are parallel to the x and y-axes. The foci are then given by (c,0) and (-c,0), while the sum of the focal radii is given as 2a. The ellipse can be drawn and labelled on the board or overhead. Students can be shown how the Pythagorean Theorem is used to determine the value of b, from $a^2 = b^2 + c^2$. Once a and b have been determined and the major axis established, the students can substitute these values into the correct form of the standard equation of an ellipse. The students can work through several examples of this type before the introduction of ellipses whose centre is (h,k) and whose axes are parallel to the x and y-axes.

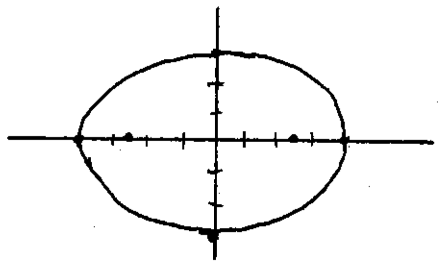
Note that translation of axes may be used to introduce the equation of the ellipse whose centre is (h,k).

Examples/Activities

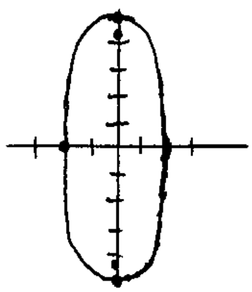
Adaptations

1. On the graphs of each of the following ellipses, label the two foci, the major and minor axes, and illustrate the total distance of the focal radii.

a)



b)



2. Given two foci (3,0) and (-3,0), and a total distance from a point P to the two foci of 10, sketch the ellipse and label the major and minor axes.

Concept B: Conic Sections

Objectives/Skills	Instructional Notes
4. Ellipse a) To define and illustrate the following terms: ellipse, foci, focal radii, major axis, minor axis, vertices, and axis of symmetry.	Students can be asked to draw the locus of a point whose total distance from two fixed points remains constant. The total distance should be chosen so that it is greater than the distance between the two fixed points, but not overly much greater. The resulting locus is an ellipse. The two fixed points are defined as the foci. The total distance is 2a, or twice the focal radii. The teacher may wish to have the students note that the general equation based on the locus is $PF_1 + PF_2 = k$. In this equation P represents any point (x,y) on the ellipse, F_1 and F_2 represent the two foci, and k the total distance of the point from the two foci. The shapes of various ellipses drawn by the students can be examined and formal definitions introduced. Major and minor axis can be identified for each ellipse and axes of symmetry can be specified also. An exercise dealing with these identifications can be assigned.

Examples/Activities

1. A manufacturer of satellite dishes buys components from various factories, and assembles them in her plant. Recently, a new factory has begun producing a new receiver that is 36 cm in length. If the satellite manufacturer wishes to use this receiver, her company will have to alter their dish design. If the transmission waves must hit the end of the receiver, what is the equation of the parabolic reflecting surface of the satellite dish?

2. You are in charge of introducing solar ovens to a society in which there is little fuel, but lots of sunlight. The parabolic reflectors have been manufactured to the specifications of the equation $y = 1/8\ (x - 4)^2$. The ovens work by collecting the sun's rays and reflecting them to the focus, where the food is placed on a small stand to be cooked. The problem is that the stands have not arrived, and you must determine the focus, and then build stands to fit. What is the focus of these ovens?

3. As a comet approaches a star, it becomes affected by the gravitational field of the star. It bends toward the star, passes it, and then slowly breaks away from the gravitational pull of the star. It is noticed that the comet's path approaching and leaving the vicinity of the star resembles a parabola. If the star represents the focus, and comet's nearest approach the vertex, what is the equation of the parabolic path the comet seems to follow if its closest approach is 3×10^5 km?

Adaptations

As an alternative approach, the teacher could introduce the parabola by using the concept of eccentricity.

Concept B: Conic Sections

Objectives/Skills

3.d)
To solve word problems involving the parabola.

Instructional Notes

Word problems could be integrated throughout all of Objective 3. It is expected that students will do word problems based on the parabola as they are introduced and follow through with such exercises on each sub-objective.

Most texts have a number of word problems which are suitable for use with these topics.

It may be of interest to the students, however, to look at some real-world situations, and attempt to approach problems from 'research team' concept. (PSVS)

As one example, the teacher may supply a bulb from a flashlight, or headlight, and ask the students to take measurements, and to predict the equation of the parabolic surface needed to make the flashlight, etc.

Note: Some students may need to be introduced to the concept "if light rays parallel to the axis strike the surface of a parabola, they are reflected through the focus". The converse of this statement is also true.

Examples/Activities

Determine the equation of the parabola given the following information:

1. The focus is (3,2) and the vertex is (3,1).
2. The focus is (4,-3) and the vertex is (4, -6).
3. The focus is (-3,2) and the vertex is (-3, 1.5).
4. The focus is (5,2) and the vertex is (5,4).
5. The focus is (-2,6) and the vertex is (-3,6).
6. The vertex is (4,1) and the directrix is y = -1.
7. The vertex is (-3,4) and the directrix is y = 7.
8. The vertex is (-2,6) and the directrix is x = -1.
9. The focus is (-4,-2) and the directrix is y = -6.
10. The focus is (6,-3) and the directrix is y = -2.
11. The focus is (7,2) and the directrix is x = 6.

Adaptations

Students can be asked to sketch the graphs of the equations in each of the exercises in the previous column. The utilization of the 'latus rectum' (the line segment through the focus and perpendicular to the segment joining the focus and the vertex) might be employed. Note that the endpoints of the latus rectum are on the parabola and that its total length is 4p units. These concepts make it quite simple to sketch the parabola from the equation written in $x^2 = 4py$ form.

Objectives/Skills

3.c)
To determine the equation of a parabola from the given data: focus and directrix, vertex and directrix, focus and vertex.

Instructional Notes

In this section, students work with the equation $x^2 = 4py$, where p is the distance from the focus to the vertex. Have students determine the distance from the focus to the vertex in several examples, from the vertex to the directrix, and from the focus to the directrix. Remember to have them determine whether these values should be positive or negative.

Once the students are able to determine the value of p, they should be asked to write the equation of the parabola in a series of exercises, using the equation $x^2 = 4py$,

or $(x - h)^2 = 4p(y - k)$.

Examples/Activities

Adaptations

Determine the equation of each of the following parabolas, analyze the equation, and sketch the graph of the equation.

1. The focus is (3,1) and the directrix is $y = -1$.
2. The focus is (4,-2) and the directrix is $y = -3$.
3. The focus is (-3,2) and the directrix is $x = -5$.

State the vertex in each of the following. Translate the axes to the vertex. Rewrite the equation using these translations. Sketch the graph of each.

1. $y + 3 = 4(x - 2)^2$
2. $y - 2 = -2(x + 3)^2$
3. $y + 6 = 1/4(x - 3)^2$
4. $2(y - 2)^2 = x + 5$
5. Write the equation of the parabola $(x - 3)^2 = 2(y + 4)$ when the axes are translated such that the translated origin is at (3, -4).

Concept B: Conic Sections

Objectives/Skills	Instructional Notes
3. Parabola a) To determine the equation of a parabola in the general form $y - k = a(x - h)^2$ or $x - h = a(y - k)^2$, given the focus and directrix.	In Mathematics A 30 the parabola was studied using the process of completing the square to obtain the equation in standard form. This process of analyzing and sketching the parabola, might be reviewed as an introduction to this section. For this objective, the teacher could start by having the students draw the locus of a point equidistant from a given point (later called the focus) and a given line (later called the directrix). A few examples of these loci could be displayed and discussed, with the definitions being introduced, and the parabola being identified as the result of this locus. The students could then be asked to determine the equation of a parabola using this definition and utilizing the distance formula (distance from a point to the focus equals the distance from that point to the directrix). A set of exercises could then be assigned so that students could practice determining such equations. Computer programs could also be utilized for this section. Note that the equation of the parabola given the focus and directrix will be developed in other ways in the following sections.
3.b) To find the equation of a parabola with its vertex at the point (h,k) by replacing x by $x^1 = x - h$ and replacing y by $y^1 = y - k$ in the equation $x^2 = 4py$.	The equation $x^2 = 4py$ is another form used in describing the parabola, where p represents the distance from the focus to the vertex. Since the shape of a parabola is not affected by the position of its vertex, but only by the coefficient 'a' in $y - k = a(x - h)^2$, it is sometimes more convenient to have a translation of the axes to obtain a simpler version of the equation. Have the students practise translating the axes to the vertex and sketching the graph at the origin of the new axes. Once this has been done, the equation can be rewritten using $x^1 = x - h$ and $y^1 = y - k$ to obtain the new equation in the form $y^1 = ax^{1\,2}$.

Examples/Activities

Adaptations

Determine the equation of each circle.

1. The centre is at (3,0) and a y-intercept is (0,4).
2. The centre is at (3,-2) and a y-intercept is at (0,-3).
3. The centre is at (-2,5) and an x-intercept is at (-6,0).
4. The endpoints of a diameter are (6,0) and (-6,0).
5. The endpoints of a diameter are (0,5) and (0,-1).
6. The endpoints of a diameter are (8,2) and (3,2).
7. The endpoints of a diameter are (4,-2) and (-2,6).
8. The endpoints of a diameter are (-3,5) and (-7,-3).
9. The centre is (0,0) and a point on the circle is (2,3).
10. The centre is (4,1) and a point on the circle is (1,-3).
11. The centre is (-6,-5) and a point on the circle is (1,-2).
12. The centre is (5,1) and a tangent line is given by $3x + y = 6$.
13. The centre is (7,8) and a tangent line is given by $2x + 3y = 12$.

Concept B: Conic Sections

Objectives/Skills

2.c)
To determine the equation of a circle in either of the forms $x^2 + y^2 = r^2$, or $(x - h)^2+(y - k)^2 = r^2$ from some given data: centre and intercepts, end points of a diameter, centre and a point on the circumference, centre and the equation of a tangent line, etc.

Instructional Notes

Students could be given a series of exercises in which they are asked to determine the equation of a circle based on information given to them. It is expected that they should use various techniques to determine the centre and radius from the given information, and then use these to write the equation. In cases where they are not able to determine a centre and radius they should conclude that it does not represent a circle, that they are lacking some information, or that they need to employ other methods to extract the information.

The students may also be asked to graph the information and to complete the analysis in each case.

Suitable real-world problems may be found and utilized in this section.

Examples/Activities

1. Write the equation of a circle in the form $Ax^2 + By^2 + Cx + Dy + E = 0$, given;

   a) Centre (2,0), r = 4
   b) Centre (-1,2), r = 2
   c) Centre (5,-2), r = 7
   d) Centre (-5, -4), r = 3
   e) Centre (0, 0), r = 5

2. What equation would describe the locus of a point 5 units from the point (3,-2)?

3. What is the equation that could be used to describe the locus of a point 11 units from the midpoint of (7,-3) and (-3, 7)?

4. What equation would describe the locus of a point 3 units from the point of intersection of $2x - 3y = -1$ and $3x + y = 15$?

1. Complete the square in each of the following;
   a) $x^2 + 6x +$___
   b) $x^2 - 12x +$____
   c) $x^2 + 5x +$____
   d) $y^2 - 7y +$____
   e) $3x^2 + 4x +$_____

2. Convert each of the following equations to the form $r^2 = (x - h)^2 + (y - k)^2$, analyze the results, and draw a sketch of the graph of the equation using your analysis.
   a) $x^2 + y^2 - 2x + 6y - 6 = 0$
   b) $x^2 + y^2 + 8x - 4y - 5 = 0$
   c) $x^2 + y^2 + 10y + 16 = 0$
   d) $x^2 + y^2 - 10x - 2y + 10 = 0$
   e) $3x^2 + 3y^2 + 12x - 6y + 3 = 0$

Adaptations

Computer programs are also available that will allow the students to study circles on their own. Many of the newer programs are interactive and allow for student experimentation, rather than simply being a tutorial book lesson. (IL) (TL)

Concept B: Conic Sections

Objectives/Skills	Instructional Notes
2. Circle a) To determine the equation of a circle in the general form **$Ax^2 + By^2 + Cx + Dy + E = 0$,** when given the centre and radius.	A review of the definitions associated with the circle may be useful. Terms such as radius, diameter, centre, chord, secant, etc., should be revisited. Students could be asked to draw the locus of a point equidistant from a given point and to discuss the result as another way to describe a circle. If the given point is equated to the centre and the equal distance to the radius, the circle can then be defined by an equation developed from the distance formula. In other words, if the given point (centre) is (h,k) and the equal distance is the radius, r, then the equation of the circle is $r^2 = (x - h)^2 + (y - k)^2$. Computer programs might also be used for learning about the conic sections, including the circle. (TL)
2.b) To change a given equation of a circle in the general form to the centre-radius form $r^2 = (x - h)^2 + (y - k)^2$ and to sketch and analyze its graph.	To introduce this section, the process of 'completing the square' can be reviewed. Once the process has been reviewed, students can utilize it in several examples of the type $x^2 + y^2 - 4x + 2y = 4$. When students have changed these equation to the form $r^2 = (x - h)^2 + (y - k)^2$, they can be instructed to identify through analysis the centre (h,k) and the radius, r. Once these have been identified for the equation, the sketch of the circle can be drawn. The teacher may wish to have the students further their analysis by identifying the domain and the range for each sketch and to note whether these graphs represent functions. (CCT)

Examples/Activities

1. What is the path of a point marked on the edge of a bicycle tire as it rolls away? (Ask if the speed of the bicycle affects the path of the point.)

2. What is the path of a satellite orbiting Earth at a distance of 600 km above the Earth's surface?

3. What is the path of a point that follows $3x + 2y = 12$ as x increases from 0 through positive values? Can you locate this point for any value of x ? y?

4. A goat is tethered to a tree 1 m in diameter. The tether is 10 m long. What is the goat's path if it continues to walk around the tree in the same direction? Explain your answer.

5. What is the locus of a point that is always equidistant from the two endpoints of a segment, if all points are in the same plane? if these points are not coplanar?

Adaptations

Students could be asked to describe loci that they have seen in real life. These might be from any subject area. The class could then attempt to draw the loci based on the students' descriptions. This exercise can be used to highlight the importance of communication in cooperative activities. (COM)

An activity such as drawing the path of a rocket to the moon may involve some research from various disciplines. In order to do this example, one needs to know that it would take over two days to travel to the moon. The earth's rotational speed and direction of rotation must be included, and the fact that the moon also moves in its orbit over the two plus days.

Appendix F
Mathematics C 30

Concept B: Conic Sections

Objectives	Instructional Notes
1. Locus a) To define and illustrate a locus in a number of situations. b) To sketch and identify a locus given its description.	The term locus is used to describe the path of a point. Some examples of loci may be given to the students based on their experiences in the real world. Once students have learned the meaning of locus, they may be given other methods of describing the path of a point. Further examples could be discussed. Students should be asked not only to draw the locus of a point, but also to describe the locus if given a drawing of its path.

Now we get to the interesting point. Have them take the right-hand half of this matrix and multiply by the constant columns of the original systems. We get

$$\begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 23/9 \\ -2/9 \end{bmatrix}$$

and

$$\begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 16/9 \\ -1/9 \end{bmatrix}$$

They should recognize their earlier solutions in these results.

Now have them do

$$\begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -6 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 5/27 & 2/9 \\ -2/27 & 1/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and talk about the inverse of a square matrix. Have them do several inverses by this row-reduction method. Finally, show them the formula for the 2x2 inverses as in C9. Emphasize that your method is general for nxn matrices whereas coming up with a formula presents a formidable task for matrices with size greater than 2x2.

Try 2x + 5y = 3, where only the right had side has changed. Ask for a matrix solution to both.

$\left[\begin{array}{cc\|c}3 & -6 & 9\\2 & 5 & 4\end{array}\right]$	$\left[\begin{array}{cc\|c}3 & -6 & 6\\2 & 5 & 3\end{array}\right]$
$\left[\begin{array}{cc\|c}1 & -2 & 3\\2 & 5 & 4\end{array}\right]$	$\left[\begin{array}{cc\|c}1 & -2 & 2\\2 & 5 & 3\end{array}\right]$
$\left[\begin{array}{cc\|c}1 & -2 & 3\\0 & 9 & -2\end{array}\right]$	$\left[\begin{array}{cc\|c}1 & -2 & 2\\0 & 9 & -1\end{array}\right]$
$\left[\begin{array}{cc\|c}1 & -2 & 3\\0 & 1 & -2/9\end{array}\right]$	$\left[\begin{array}{cc\|c}1 & -2 & 2\\0 & 1 & -1/9\end{array}\right]$
$\left[\begin{array}{cc\|c}1 & 0 & 23/9\\0 & 1 & -2/9\end{array}\right]$	$\left[\begin{array}{cc\|c}1 & 0 & 16/9\\0 & 1 & -1/9\end{array}\right]$

Point out the obvious duplication effort. Ask students to consider a family of half a dozen such systems (rather than two) where the only difference is the constant column. Obviously there is a huge duplication of effort involved to solve them all. This is one very good reason why we need the concept of a matrix inverse! Now ask them to solve
3x - 6y = 1,
2x + 5y = 0,
and
3x - 6y = 0.

Try 2x + 5y = 1, but suggest they do it both at once, as

$$\left[\begin{array}{cc|cc}3 & -6 & 1 & 0\\2 & 5 & 0 & 1\end{array}\right]=\left[\begin{array}{cc|cc}1 & -2 & 1/3 & 0\\2 & 5 & 0 & 1\end{array}\right]=\left[\begin{array}{cc|cc}1 & -2 & 1/3 & 0\\0 & 9 & -2/3 & 1\end{array}\right]=\left[\begin{array}{cc|cc}1 & -2 & 1/3 & 0\\0 & 1 & -2/27 & 1/9\end{array}\right]=\left[\begin{array}{cc|cc}1 & 0 & 5/27 & 2/9\\0 & 1 & -2/27 & 1/9\end{array}\right]$$

Appendix E

Matrices

Take a simple problem with two equations and two unknowns and solve in the usual manner. Introduce a matrix solely as a notational convenience, doing the work in a parallel fashion and motivate the defining of the matrix this way.

3x - 6y = 9 2x + 5y = 4	$\left[\begin{array}{cc\|c} 3 & -6 & 9 \\ 2 & 5 & 4 \end{array}\right]$
multiply eqn 1 by 1/3	(we have an 'x', a 'y', and a constants column)
x - 2y = 3 2x + 5y = 4	$\left[\begin{array}{cc\|c} 1 & -2 & 3 \\ 2 & 5 & 4 \end{array}\right]$
take two of new eqn 1 from eqn 2	
x - 2y = 3 0x + 9y = -2	$\left[\begin{array}{cc\|c} 1 & -2 & 3 \\ 0 & 9 & -2 \end{array}\right]$
multiply new eqn 2 by 1/9	
x - 2y = 3 0x + 1y = -2/9	$\left[\begin{array}{cc\|c} 1 & -2 & 3 \\ 0 & 9 & -2/9 \end{array}\right]$
add two new eqn 2' to new eqn 1	
x - 0y = 23/9 0x + 1y = 2/9	$\left[\begin{array}{cc\|c} 1 & 0 & 23/9 \\ 0 & 1 & -2/9 \end{array}\right]$
E.g.: x = 23/9 and y = -2/9	now read this as x = 23/9 and y = -2/9

We have solved the system twice in exactly the same way but on the right we have avoided about 10 equal signs and about the same number of +'s. This is simply a good notation that speeds up the work. Define this rectangular array as a matrix. Solve a bunch of systems this way to that students get used to the operations on the matrix. These are exactly the row operations you talk about later on p. 641. They are not confusing when related to something students know already like the solving of systems of equations.

If you like, now include sections such as C2 to C7. Now introduce, as C8 maybe, the problem of simultaneously solving families of systems of equations; such as,
3x - 6y = 9,
2x + 5y = 4, and
3x - 6y = 6.

Lesson 1: The Need For Proof (CCT, NUM)

Our intuition often serves us well, but there are times when it will mislead us. Consider the following problems and see how well your intuition serves you.

Problem 1: A motorist drove the 300 km from Saskatoon to Meadow Lake at a speed of 100 km/h. Poor visibility caused the motorist to make the return trip at 80 km/h. What was the motorist's average speed for the trip?

(a) What is your intuitive answer to the question? ________

(b) Let's check it out:

How long does it take to drive there?
How long does it take to drive back?
What is the total time spent driving?
What is the total distance travelled?
What then is the average speed?

If the total trip takes 6.75 hours, and if the average speed had been 90 km/h, you would have covered a distance of ________ km. It is clear that the intuitive answer of 90 km/h is too _______.

Problem 2: Two containers, one holding a litre (1000 mL) of cola, the other holding a litre of coffee, are standing beside one another. A cup (250 mL) of cola is transferred to the coffee container and thoroughly mixed in with the coffee. A cup of the coffee-cola mix is then transferred back to the cola container. Is there more coffee in the cola container or is there more cola in the coffee container?

(a) What is your intuitive answer?

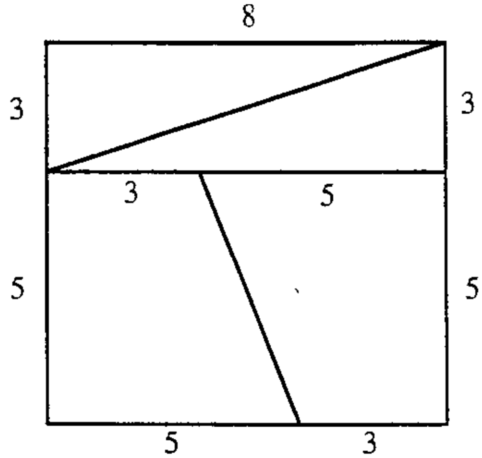
(b) Let's check it out.

	Cola Container		Coffee Container	
Action Taken	Amount Cola	Amount Coffee	Amount Cola	Amount Coffee
original situation				
1 cup from cola container to coffee container				
1 cup from coffee container to cola container				

As you may have realized from the above examples, intuition needs to be tested by investigating situations in a precise manner.

Consider the problem below. Your intuition will tell you something is amiss. See if you can determine what that is.

Problem 3: On a sheet of graph paper draw the figure shown below.



(a) What is the area of the square above?

(b) From your sheet of graph paper, cut out the four pieces shown in the figure and rearrange them to form a rectangle. (You do not have to flip over any of the four pieces.)

(c) What is the area of the rectangle formed in part (b)?

(d) What is the resulting contradiction?

(e) Explain why this contradiction arises?

Assignment

1. The distance from Regina to Saskatoon is approximately 250 km. If road conditions are such that a motorist can only travel 50 km/h on the trip to Saskatoon, at what speed must the motorist drive on the return trip in order to average 100 km/h for the round trip?

2. If it were possible to wrap the earth with a metal ring at its equator, you would need a ring whose circumference was approximately 40 000 km. Suppose you inserted an extra 2 m (0.002 km) into the ring so that it is now 40 000.002 km in length. This ring would no longer be snug against the earth. Do you think there would be enough room for you to crawl between the earth and the ring? Check up on your intuitive answer. (Hint: $C = \pi d$, use the value of π found on your calculator.)

3. There are many different versions of the following old mathematical riddle. A wealthy man at his death left his stable of seventeen beautiful horses to his three sons. He specified that the eldest was to have one-half of the horses, the next one-third, and the youngest one-ninth. The three young heirs were in despair, for they obviously could not divide seventeen horses this way without calling in the butcher. They finally sought the advice of an old and wise friend, who promised to help them. He arrived at the stable the next day, leading one of his own horses. This he added to the seventeen and directed the brothers to make their choices. The eldest took one-half of the eighteen, or nine; the next, one-third of the eighteen, or six; the youngest took one-ninth of the eighteen or two. When all seventeen of the original horses had been chosen, the old man took his own horse and departed. What's the catch in this story?

4. Three fuel saving devices were invented in the same year. One claimed to reduce fuel consumption by 10%, a second claimed to reduce fuel consumption by 40%, and a third claimed to reduce fuel consumption by 50%. A motorist decided to attach all three devices to his motor to really save on fuel.
   (a) Intuitively, you are lead to believe the motorist would have a 100% saving on fuel. Why is that impossible?
   (b) What will be the actual percentage saved on fuel?

5. (a) On a piece of graph paper, draw a figure having the dimensions shown at left below.

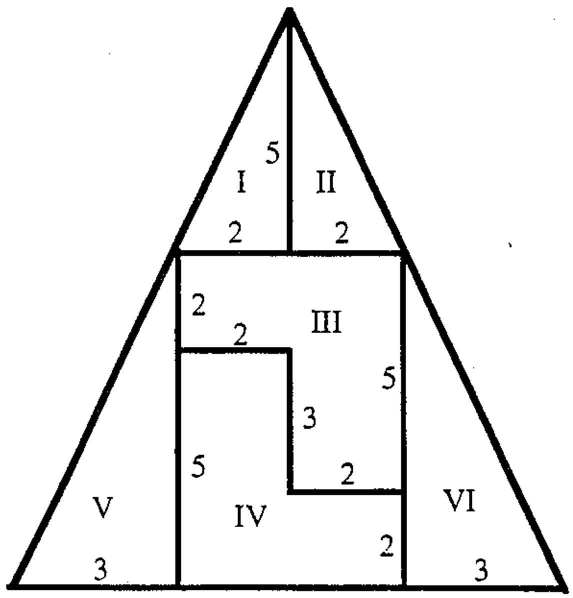


figure 1

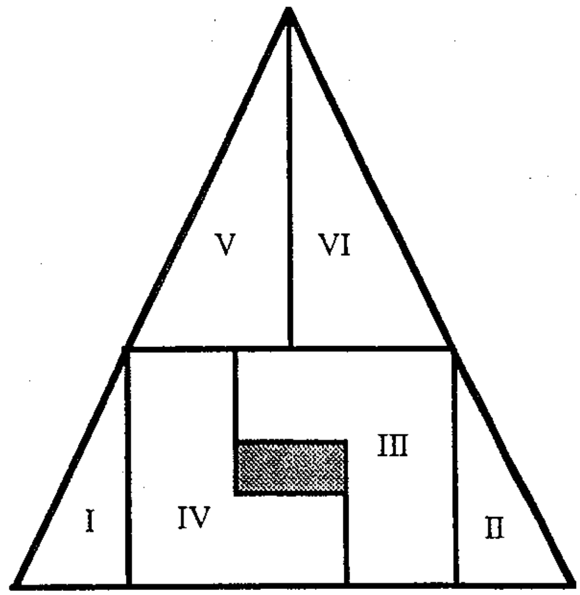


figure 2

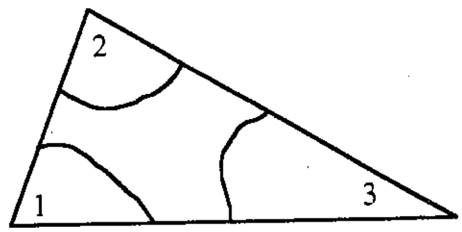
(b) Measure the base and altitude of the triangle in figure 1 and determine its area.
(c) Cut out the six pieces of the triangle from your piece of graph paper and rearrange them as shown in figure 2.
(d) Measure the base and altitude of the triangle in figure 2 and determine its area.
(e) Where did the paper disappear to upon rearranging the six pieces?

Lesson 2: Inductive Reasoning (CCT, IL)

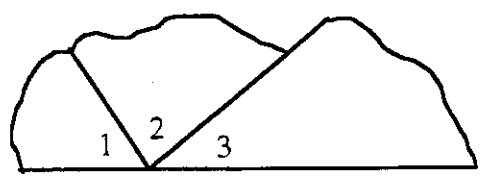
When we collect evidence, observe patterns, and draw conclusions from these observed patterns, we are using a reasoning process that is termed **inductive reasoning.** The scientist is a prime user of inductive reasoning, as observations lead to discoveries of regularities which in turn lead to the theories of the laws of nature.

An example of inductive reasoning that you may have already tried in Mathematics 10 follows.

On a piece of paper draw any triangle. Use scissors to cut out the triangle.



Then rip out each of the three angles of the triangle and rearrange the three angles as shown below. The three angles appear to have a sum of 180 degrees.



As other class members perform the experiment, having initially drawn their own unique triangle, the evidence mounts that the three angles in a triangle have a sum of 180 degrees. It is important to point out that gathering this evidence does not prove the conclusion that is reached. It merely suggests that conclusion. You might find it of interest to try and recall how you proved that the sum of the angles of a triangle is 180 degrees.

Example 1:

(a) Draw any quadrilateral in the space at right.
(b) Determine the midpoints of each side of the quadrilateral.
(c) Connect these midpoints with line segments.
(d) What kind of a quadrilateral seems to result?
(e) Either repeat your experiment several times or compare your result with those of other class members.
(f) What do you conclude?
(g) What kind of reasoning has lead you to your conclusion?

Example 2:

(a) Complete each of the following calculations:
1 x 8 + 1 =
12 x 8 + 2 =
123 x 8 + 3 =
1234 x 8 + 4 =

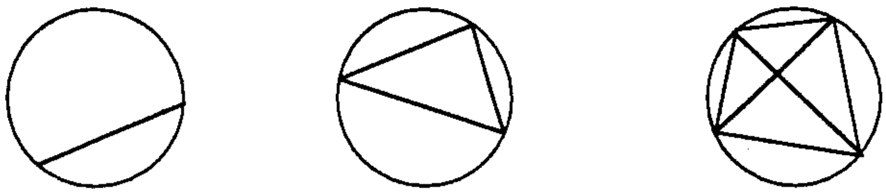
(b) Based on your calculations above, predict the answers to the following calculations.
12345 x 8 + 5 =
123456 x 8 + 6 =
123456789 x 8 + 9 =

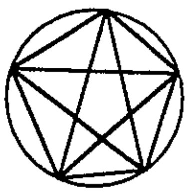
(c) Use a calculator to check your predictions in part (b).

(d) Does this pattern continue?

Example 3:

Consider the pattern suggested by the following figures.



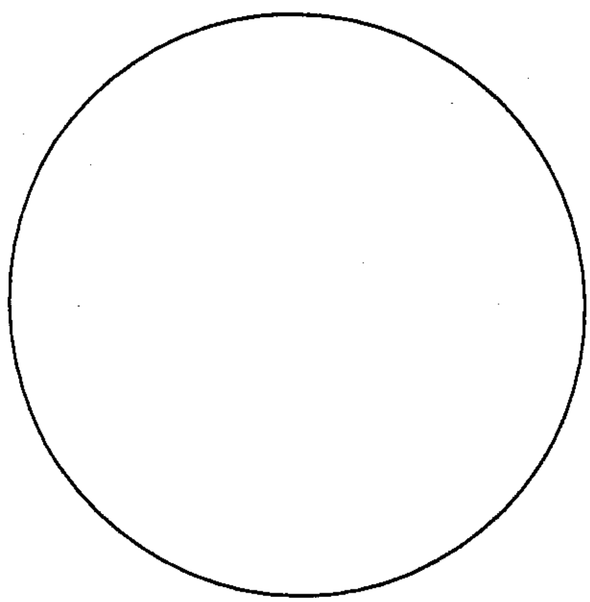


Number of points connected	2	3	4	5	6	
Number of resulting regions	2					

(a) Count the number of regions resulting when 3, 4, and 5 points are connected and complete the appropriate portion of the table.

(b) Based on the data collected, predict how many regions can be formed when 6 points are connected.

(c) Place 6 points on the circumference of the circle at the top of the next page and connect them in every possible way. Count the number of resulting regions.



(d) What does this example illustrate about inductive reasoning?

Inductive reasoning contains two key steps.

Step 1: Observe that for every case checked, a certain property is true.
Step 2: Generalize that the property is true for all cases.

There is a caution however. Your generalization in step 2 may not be correct since you have not examined all possible cases in step 1.

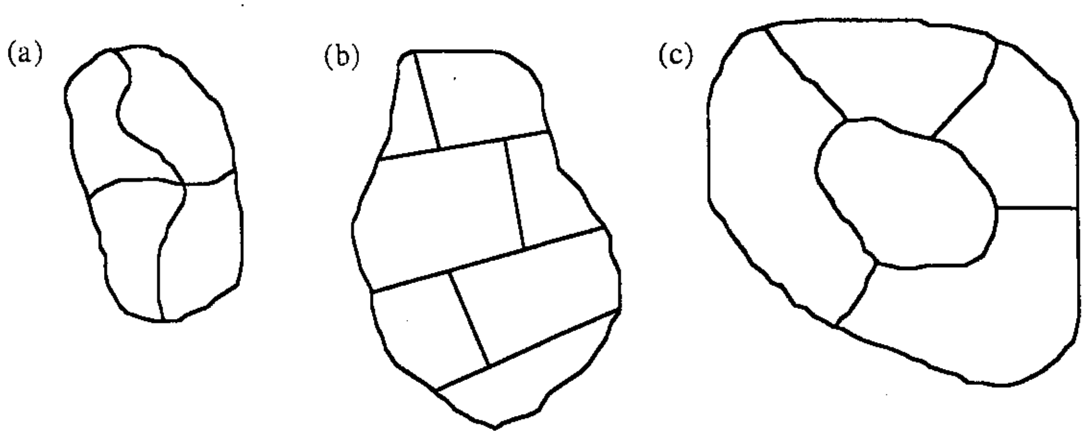
Assignment

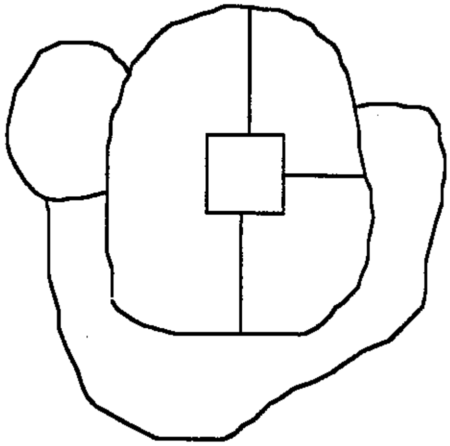
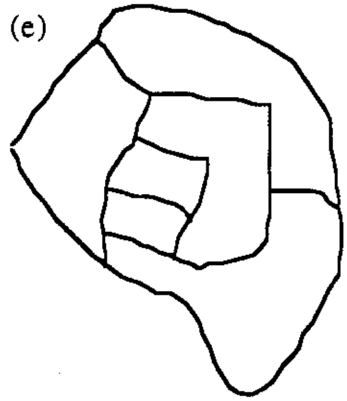
1. (a) Consider the number patterns shown below and verify that they are correct.

$1 =$	1^2
$1 + 3 =$	2^2
$1 + 3 + 5 =$	3^2
$1 + 3 + 5 + 7 =$	4^2
$1 + 3 + 5 + 7 + 9 =$	5^2

(b) What are the next two lines in the number pattern if it is continued? Verify these.
(c) According to the pattern above, what would be the sum of the first ten positive odd integers?
(d) Verify your result in (c) by actually adding $1 + 3 + 5 + 7 + \ldots + 17 + 19$.
(e) Do you think the pattern above continues indefinitely?
(f) What kind of reasoning are you using in this problem?

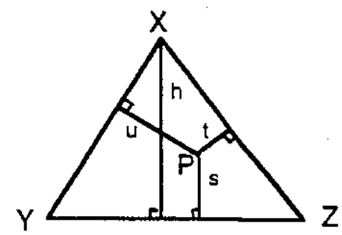
2. Suppose you are required to color a map of a continent whose countries are shown below. You must color the map using as few colors as possible, but no two countries sharing a border are to be the same color. Determine the minimum number of colors required for each map.



(d)  (e) 

(f) Draw a map of a continent and its countries that requires more than four colors in order to distinguish the countries.
(g) Inductively, what conclusion can be reached about map coloring?
(h) Does this prove that a map does not exist for which more than four colors are needed?

Remark: It was not until 1976 that two Americans, with the aid of a computer, proved that maps requiring more than four colors do not exist. You can read about it in the October 1977 issue of the magazine *Scientific American*.



3.
(a) Draw a fairly large equilateral triangle XYZ.
(b) Draw the altitude from X to YZ.
(c) Choose any point P inside the triangle.
(d) Draw perpendiculars from P to the sides of the triangle.
(e) Measure h, s, t, and u, the lengths of the altitude, and the three perpendiculars respectively, to the nearest mm.
(f) Repeat parts (c), (d), and (e) as many times as is necessary until you can state a generalization concerning h, s, t, and u.
(g) Do your experiments prove that your generalization is true?

4.
(a) Complete each of the following statements.

23 x 64 =	26 x 93 =	41 x 28 =
32 x 46 =	62 x 39 =	14 x 82 =
69 x 64 =	84 x 36 =	
96 x 46 =	48 x 63 =	

(b) Based on the pattern above one might be inclined to generalize that the product of a pair of two-digit numbers is the same as the product of the numbers formed by reversing their digits. Is this generalization true for all two digit numbers? Try a few of your own choosing.

(c) For what kinds of pairs of two digit numbers does the generalization in part (b) seem to hold? Search your data in part (a) carefully.

5. Consider the function f(x) and $x^2 + x + 41$. The table below suggests that whenever x is replaced by a positive integer, f(x) is a prime number. Certainly that is not the case for the function $g(x) = x^2 + x$ because g(2) = 6 and 6 is not prime.

x	1	2	3	4	5	6	7
f(x)	43	47	53	61	71	83	97

(a) Continue the table begun for f(x) until you reach the first value of x for which f(x) is not prime. What is the x value?
(b) What conclusion might inductive reasoning have lead you to had you not done part (a) of this question?
(c) What then is the value of inductive reasoning?

6. Shown below are the first several rows of Pascal's triangle. The sum of the numbers in each row is shown at right. According to the pattern, what would be the sum of the numbers in the 20th row of Pascal's triangle?

1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64

Lesson 3: Deductive Reasoning (CCT, NUM, IL)

Activity

The following is a game for two players.

Place a pile of 20 toothpicks on your desk. Determine the starting player. Players alternate turns removing 1 or 2 toothpicks per turn from the pile. The player to remove the last toothpick is the winner.

Play this game several times until you hit upon a winning strategy. If you wish to discover the winning strategy more quickly, you could start the game with fewer than 20 toothpicks.

Deductive reasoning allows us to draw conclusions using logic that is based on information we accept as true.

In reference to the game above, let us number the toothpicks from 20 to 1 as shown below.

20	19	18	17	16	15	14	13	12	11
10	9	8	7	6	5	4	3	2	1

Player A will win the game by taking the last toothpick--toothpick #1 according to the definition of winning in the rules. This will be possible for player A provided he/she can take toothpick number 4. If player A takes #4, then player B is forced to take toothpick #3 or toothpicks #2 and #3. If B takes #3, A can take both #2 and #1 and thus win. Similarly if B takes #3 and #2, then A can take #1 and win. Thus, using logic, we can see that player A can win by taking toothpick #4. What will guarantee that player A will be able to take #4? Consider what happens if player A takes #7. Then player B is forced to take #6 and #5 or just #6. If B takes #6 and #5, A can take #4 and win. If B takes #6, A can take #5 and #4 and win. Continuing this kind of logical reasoning, one can see that player A will win by taking #1, which means player A must take #4, which means player A must take #7, #10, #13, #16, and #19. Thus if player A starts the game he/she should take toothpicks 20 and 19. If A does not start the game, there is no way A can win if B already knows the winning strategy. If B does not know the strategy, A could still win by being sure to take one of the winning intermediate numbers as B allows A to do so.

Example 1:
Try the following number trick. Complete the chart using three different starting numbers. The first has been done for you.

Directions	1st #	2nd #	3rd #
Choose any number	11		
Multiply by 4	44		
Add 10	54		
Divide by 2	27		
Subtract 5	22		
Divide by 2	11		
Add 3	14		
Subtract the original number	3		

Inductive reasoning would suggest to us that the result would always be 3. We can use deductive reasoning to prove what inductive reasoning suggests. If we let our starting number be x, then the step by step results are shown below. Complete the chart.

If we accept the polynomial manipulations to the right of each statement as being correct, then the conclusion that we reach, namely that 3 is always the result, is also true. Once again, reasoning in this manner is termed deductive reasoning.

Directions	1st #
Choose any number	x
Multiply by 4	$4x$
Add 10	$4x + 10$
Divide by 2	
Subtract 5	
Divide by 2	
Add 3	
Subtract the original number	

Example 2:

(a) Choose any two positive integers.
(b) Find the square of the sum of the integers chosen in (a).
(c) Find the sum of the squares of the integers chosen in (a).
(d) How does your answer to (b) compare in size to your answer in (d)?
(e) Use deductive reasoning to prove that your conclusion in (d) will hold for any two positive integers.

Example 3:

For any two positive numbers x and y, prove that if $x > y$, then $1/x < 1/y$.

Assignment

1. Replay the game outlined at the beginning of this lesson changing the rules so that 1, 2, or 3 toothpicks can be picked up per turn. Use deductive reasoning to determine the winning strategy before you begin to play. What is the winning strategy?

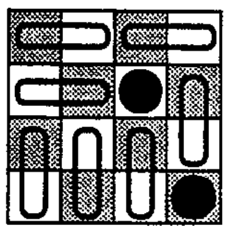
2. Make up a number trick of five or more steps (see example 1) that always results in a final value of 6.

3. A number trick appearing in the August 1973 issue of Scientific American concerns a matchbook. From an unused matchbook containing exactly 20 matches, tear out from 1 to 9 matches and throw them away. Count the number of remaining matches. Add the two digits of this number and tear out this many additional matches from the book. Tear out two more matches.
   (a) Try this number trick several times. (You do not really need to use a matchbook.) What is the result?
   (b) Prove that this trick will always work.

4. Prove that the function $f(x) = x^2 + 8x + 7$ will never yield a prime number if x is replaced by a positive integer.

5. A drawer contains an equal number of identical red socks and identical white socks. If you reach into the drawer and are unable to see the color of the socks, how many socks will you have to remove from the drawer to guarantee that you will have:
   (a) one pair of socks? Explain.
   (b) two pairs of socks? Explain.
   (c) x pairs of socks? Explain.

6. On a 4 by 4 checker board place two coins in any two of the squares. Your opponent in this game has 7 paper clips each of which is large enough to cover two squares. You win the game if you can place the two coins in such a way that your opponent cannot cover the remaining 14 squares with the 7 paper clips. Paper clips may only be placed horizontally or vertically and they may not overlap one another. Find a strategy so that the person who places down the two coins will always win. Use deductive reasoning to prove your result.



7. Two fathers and two sons left town reducing the town's population by only three. How can this be?

8. A bottle and a cork together cost $1.06. The bottle cost $1 more than the cork. How much does the cork cost? Prove your result.

9. If 3 cats can catch 3 mice in 3 minutes, how long will it take 100 cats to catch 100 mice? Prove your result.

10. Assuming both players to be intelligent, who started this game of X's and O's, player X or player O? Explain.

		O
	X	O
		X

Lesson 4: Conditional Statements and Proofs By Counterexample (CCT, IL)

The following statements are examples of what are called **if-then statements.**

If you cheer for the Oilers, **then** you're a winner.
If two parallel lines are cut by a transversal, **then** the same side interior angles are supplementary.
If you are honest, **then** you do not steal.

Every if-then statement has the basic form **if p, then q. p** is termed the **hypothesis** of the if-then statement, while **q** is termed the **conclusion.**

Example 1:
Identify the hypothesis and conclusion in the statement "If the Blades win five in a row, then I'll buy a season's ticket."

If-then statements are sometimes called **conditional statements** or just **conditionals**. Conditionals may not always appear in the if p then q form, but they can always be rewritten in that form.

Example 2:
Rewrite each of the following statements in the if p then q form.

(a) All parallelograms have opposite sides that are congruent.
(b) The square of an even integer is even.
(c) Eating candy will cause your teeth to decay.

A conditional statement can be proven to be false if an example can be found for which the hypothesis is true but the conclusion is false. Such an example is called a **counter example.** It only takes one counter example to prove that a conditional statement is false.

Example 3:
Provide a counter example to prove that each statement below is false.

(a) If $x > 0$, then $\sqrt{x} < X$.
(b) If the diagonals of a quadrilateral are perpendicular, then the quadrilateral is a square.

When you interchange the conclusion and the hypothesis of an if-then statement, you form the **converse** of the statement.

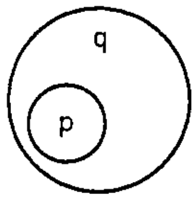
Original statement: if p, then q
Converse statement: if q, then p

Example 4:
Write the converse of each of the following if-then statements. Determine if the converse is true or false.

(a) If a triangle is equilateral, then it is acute.
(b) If a quadrilateral has four right angles, then it is a rectangle.

Example 4 shows us that the converse of a true if-then statement may be true or it may be false. By using Venn diagrams we can determine whether conclusions reached in an argument are valid. The statement "if p then q" can be illustrated by the following diagram.

Consider the true statement "If a figure is a square, then it is a polygon". The small circle p in the diagram below represents all figures that are squares. The larger circle, labelled q, represents all figures that are polygons. Clearly the relationship of the two circles is correct because all square figures are also polygons, so the little circle belongs in the big one.



Example 5:
Examine each of the following statements in light of the Venn diagram and determine the conclusion, if any, that can be reached.

(a) If a figure is a square, then it is a polygon. The figure I am looking at is a square.
(b) If a figure is a square, then it is a polygon. I am looking at a polygon.
(c) If a figure is a square, then it is a polygon. I am not looking at a square.
(d) If a figure is a square, then it is a polygon. I am not looking at a polygon.

Assignment

For each of the following statements write the hypothesis, conclusion, and converse. Determine if the converse is true or false.

1. If there's a will, then there's a way.
2. If $-5x + 1 \le 16$, then $x > -3$.
3. If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, then $\angle BAC$ is a straight angle.
4. If you are registered in a mathematics class, then you will need a calculator.

Suppose that each of the following conditionals is true. Assume also that the second statement given in (a), (b), and (c) is also true. What conclusion, if any, follows from the two statements? Use a Venn diagram to help you with your conclusions.

5. If a car has anti-lock brakes, then it must be relatively new.
   (a) This car is relatively new.
   (b) This car does not have anti-lock brakes.
   (c) This car is not new.

6. If you are my student, then you love math.
   (a) You hate math.
   (b) You love math.
   (c) You are not my student.

7. If it rains tomorrow, I'll pick you up for school.
   (a) It rains tomorrow.
   (b) I don't pick you up for school.
   (c) It does not rain tomorrow.
   (d) I pick you up for school.

8. If you own a Saturn, then you own a car.
   (a) You do not own a car.
   (b) You own a Honda.
   (c) You own a car.

Give a counterexample to disprove each of the following statements.

9. All prime numbers are odd.

10. If $a^2 = b^2$, then a = b.

11. $\frac{a}{b} < \frac{a+1}{b+1}$ any positive integers a and b.

12. $\frac{j}{k} < \frac{2j}{k}$ for any real non-zero numbers j and k.

13. There is a prime number between any pair of consecutive multiples of 5.

The following four problems are puzzles that appear in many math puzzle books in one form or another. These were taken from a book entitled *Fun with Puzzles* written by Joseph Leeming and published by Scholastic Book Services in 1971. By using deductive reasoning and by rewording statements in the if-then form, see if you can solve them.

14. A man owned a fox, a duck, and a bag of corn. One day he was on the bank of a river, where there was a boat only large enough for him to cross with one of his possessions. If he left the fox and duck alone, the fox would eat the duck. If he left the duck and the corn alone, the duck would eat the corn. How did he get safely across the river with all three of his possessions intact?

15. A man went to town one day with $5 in his pocket, but returned in the evening with $15. He bought a hat at the men's furnishings store and some meat at the meat market. Then he had his eyes tested for glasses. Now, this man got paid every Thursday by check, and the banks in the town are open on Tuesday, Friday, and Saturday only. The eye doctor does not keep his office open on Saturday, and the meat market is not open on Thursday or Friday. What day did the man go to town?

16. Three small boys were talking together when they were joined by an older man. The newcomer noticed that each of the three boys had a smudge of dirt on his forehead. "Boys," he said, "will each of you look at the foreheads of the other two and if you see a smudge of dirt on either or both, raise your hand." All three boys looked and all three raised their hands. "Now," said the old gentleman, "if one of you is certain that he has dirt on his own forehead and can tell me how and why he knows this, he is to raise his hand and I will give him a quarter." The three boys looked at each other for a few moments, and one of them suddenly raised his hand. Can you figure out how he knew that he had a smudge on his forehead?

17. Mr. Jones one day got off the train in Chicago and while passing through the station met a friend he had not seen in years. With his friend was a little girl. "Well, I certainly am glad to see you," said Mr. Jones. "Same here," said his friend. "Since I last saw you I've been married--to someone you never knew. This is my little girl." "I'm glad to meet you," said Mr. Jones. "What's your name? "It's the same as my mother's," answered the little girl. "Oh, then your name is Anne," said Mr. Jones. How did he know?

18. Here's another classical problem.
A census taker came to a house and asked the mother for the ages of her three daughters. The mother, being a mathematician, told him that their ages multiplied together were 36 and their sum was her house number. The census taker, not being a mathematical slouch himself, looked at the house number and quickly told the mother that she had not given him enough information to determine their ages. The mother responded, "Oh, I forgot to tell you that the oldest girl doesn't like chocolate pudding." Given this information, the census taker was able to calculate their ages. How old were each of the girls?

Lesson 5: Integer Property Proofs (CCT, NUM)

Every integer is either even or odd.

If an integer is even, it can be written in the form 2b where b itself is an integer. Thus 54 is even because it can be written as 2(27) and 27 is an integer. Thus 97 is not even because 97 = 2(48.5) and 48.5 is not an integer.

If an integer is odd, it can be written in the form 2c + 1 where c is an integer. Thus 97 is odd because it can be written as 2(48) + 1 and 48 is an integer.

If an integer is a multiple of 3 (divisible by 3) it can be written in the form 3k where k is an integer. For example 75 is a multiple of 3 because it can be written as 3(25) and 25 is an integer. Similarly, if an integer is divisible by 10, it can be written in the form 10w where w is an integer.

In general, the integer p is a multiple of the integer q if there exists an integer r such that p = rq.

Any positive two digit number having a ten's digits of t and a unit's digit of u can be written in the form 10t + u. For example 74 can be written as 10(7) + 4.

If the digits of the number described in the above paragraph are reversed, the new number formed will of the form 10u + t. For example, reversing the digits of the number 74 gives 47 which is 10(4) + 7.

Armed with the information contained in the above paragraphs, we can begin to prove several properties about integers.

Example 1: Prove that the product of any two even integers is even.

Example 2: Prove that the square of any odd integer is odd.

Example 3: Prove that the sum of a two digit number and the number formed by reversing its digits will always be divisible by 11.

Assignment
Prove each of the following statements.

1. The sum of any two odd integers is even.
2. The sum of an odd integer and an even integer is odd.
3. The product of any two odd integers is odd.
4. The product of two consecutive integers is even.
5. The sum of any three consecutive integers is a multiple of three.
6. Any common factor of two given numbers is a factor of their sum.
7. The square of any even integer is even.
8. The difference of the squares of two consecutive integers is odd.
9. The sum of the squares of two consecutive odd integers is even.
10. The square of a two digit number ending in 5 can always be formed by multiplying the ten's digit by the next larger digit and "attaching" 25. (For example 75^2: (7)(8) = 56. Attach 25 to 56 and you have 5625.)
11. The difference between a two digit number and the number formed by reversing its digits is divisible by 9.
12. The square of any integer is either divisible by 4 or leaves a remainder of 1 when divided by 4.

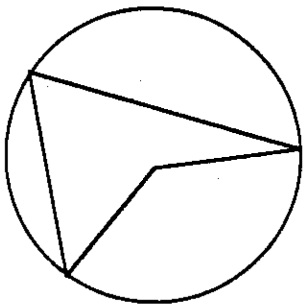
Lesson 6: Deductive Geometric Proofs and Indirect Proofs (CCT, IL)

The following four examples illustrate two column deductive geometric proofs similar to the kind of proofs in the Mathematics 20 course.

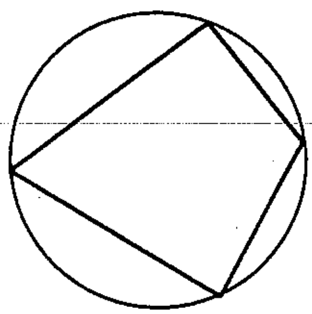
Example 1: Prove that the pairs of vertical angles formed by two intersecting lines are congruent.

Example 2: Prove that the measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

Example 3: Prove that the angle at the circumference of a circle is equal to one-half the angle at the center of the circle subtended by the same arc.



Example 4: Prove that the opposite angles of a quadrilateral inscribed in a circle (a cyclic quadrilateral) are supplementary.



Indirect reasoning is often used in order to complete a proof. Suppose that you and your friend arrive at a concert hall only to find that the two of you and the custodian are the only ones there. You are convinced that the concert must not be today. Your reasoning would go something like this. "If this were the day of the concert, there'd be hundreds of people here. We are the only ones here. Therefore this cannot be the concert day."

In an **indirect proof** we assume that the statement we wish to prove true is actually false. With this assumption we continue our reasoning until we reach a conclusion that contradicts something we know to be true. This tells us that our assumption was false and therefore the opposite of our assumption is true. In the case where there are only two possibilities, the proof is complete.

Suppose that when the principal went out to her car in the staff parking lot on Thursday at 12:45 p.m., she noticed that she had four flat tires. All of her tires were fine at 11:00 a.m. when she returned from a meeting with the superintendent. She has accused your best friend Sally of this misdemeanor. If you wanted to prove that Sally did flatten the principal's tires, you could begin your proof by assuming that Sally did flatten the tires. If she did flatten the tires, there would have had to have been a time between 11:00 a.m. and 12:45 p.m. where Sally had access to the principal's car. The curling coach, Mr. Hogline, knows that Sally was on time for his third period class, which started at 10:50 a.m., and that Sally left his class, going directly with him and the other team members, to the curling rink for a noon hour practice. Sally and the team returned to school at 12:55 p.m.with Mr. Hogline. If Sally flattened the tires, she had to have been at the car. There is evidence that she was not at the car. Therefore Sally did not flatten the tires.

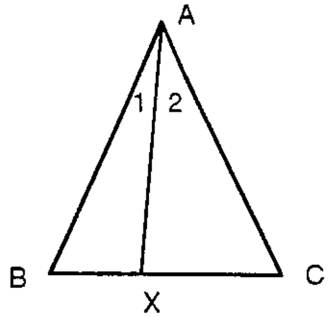
The following proofs illustrate the indirect proof process.

Example 5:
Given:

$\angle 1 \neq \angle 2$, $\overline{AB} \cong \overline{AC}$

Prove:

$\overline{BX} \ncong \overline{CX}$



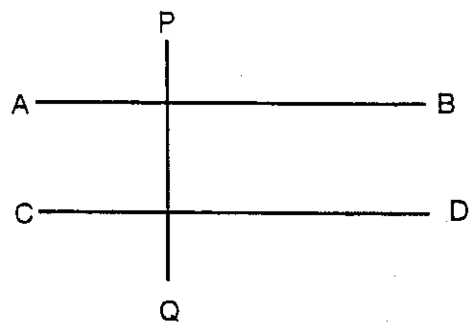
Example 6:

Given:

$\overline{AB} \perp \overline{PQ}$, $\overline{CD} \perp \overline{PQ}$

Prove:

$\overline{AB} \parallel \overline{CD}$



Example 7: Prove that $\sqrt{2}$ is an irrational number.

Example 8: To prevent Al, Betty, and Cory from dominating the High Rollers Social Club, the membership of the executive committee of that club was restricted by the conditions below. Prove that Betty is not a member of the executive.

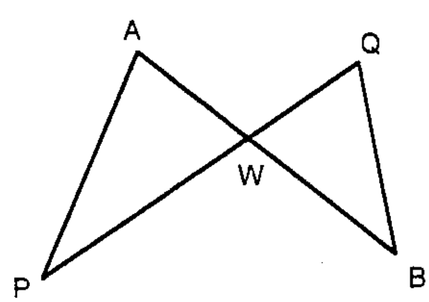
(a) If Al is a member, then Betty is not a member.
(b) If Al is not a member, then Cory is a member.
(c) If Betty is a member, then Cory is not a member.

Example 9: Prove that there are infinitely many prime numbers.

Assignment

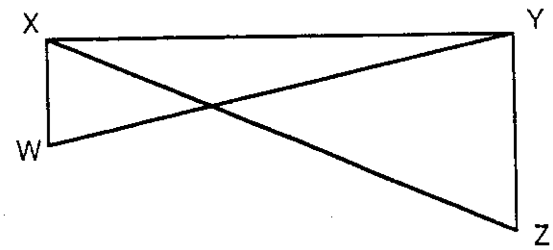
1. Given: $\overline{PQ}$ bisects $\overline{AB}$, $\overline{AP} \ncong \overline{BQ}$

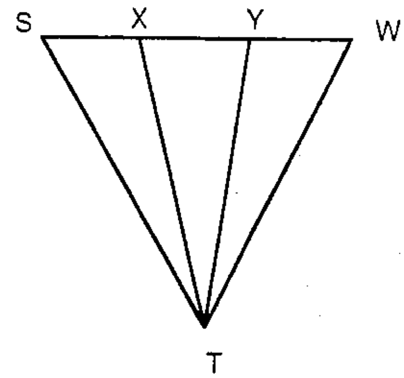
   Prove $\overline{PW} \ncong \overline{QW}$



2. Given: $\overline{WX} \perp \overline{XY}$, $\overline{ZY} \perp \overline{XY}$, $\overline{YW} \not\cong \overline{ZY}$

   Prove: $\overline{WX} \not\cong \overline{ZY}$





questions 3 and 4

3. Given: $\triangle STW$, $\overline{ST} \cong \overline{WT}$, $\overline{TX} \not\cong \overline{TY}$

   Prove: $\overline{SX} \not\cong \overline{WY}$

4. Given: $\triangle STW$, $\overline{ST} \cong \overline{WT}$, $\angle TXY \not\cong \angle TYX$

   Prove: $\overline{SX} \not\cong \overline{WY}$

5. Prove that there is no largest integer.
6. Prove that $\sqrt{3}$ is an irrational number.
7. Use an indirect proof to prove that the product of two even integers is even.

Lesson 7: Coordinate Geometry Proofs (CCT, IL, NUM)

Many proofs that take several steps to accomplish in a two column or paragraph proof can be much more easily proven using coordinate geometry. The results from earlier courses that we may need to use in our proofs are summarized below.

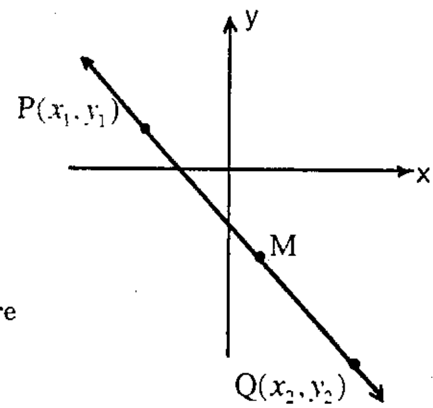
The slope of PQ is $\frac{y_1 - y_2}{x_1 - x_2}$

The length of PQ is

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

The coordinates of M, the midpoint of PQ, are

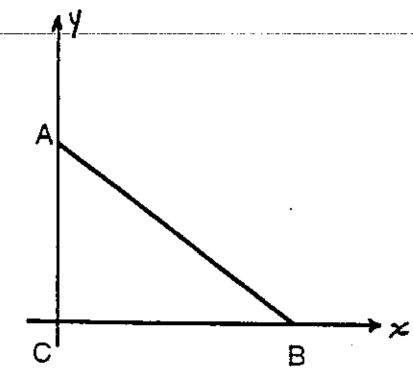
$$\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right]$$

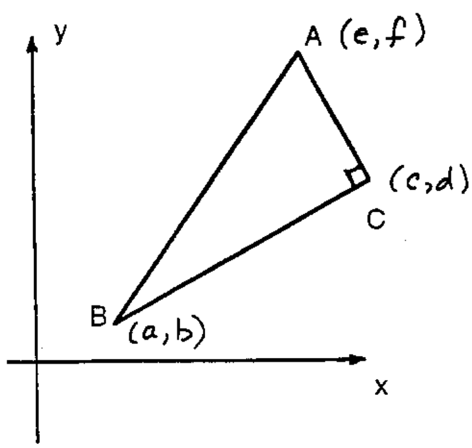


The equation of the line through $P(x_1,y_1)$ with slope m is $y - y_1 = m(x - x_1)$

Slopes of perpendicular lines are negative reciprocals of each other. This means they have a product of -1. Parallel lines have equal slopes.

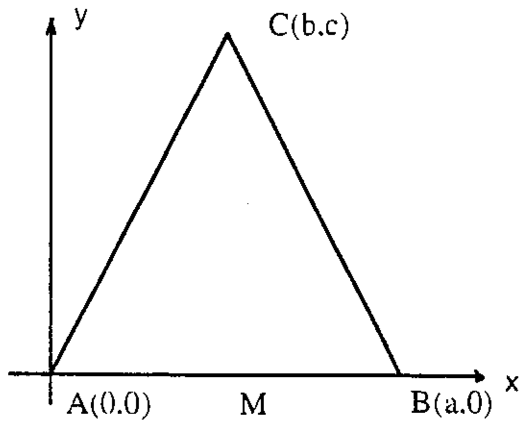
Example 1:
Use coordinate geometry to prove that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.

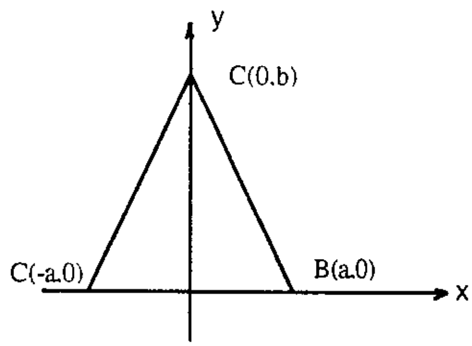




Remarks: The most important step in a coordinate geometry proof is placing the figure onto the coordinate plane in such a way as to make the mathematical manipulations as simple as possible. Clearly had we chosen the coordinates of the vertices of the right triangle as shown at the left, our calculations would have been much more involved. We would have had to use the fact that AC and BC have slopes that are negative reciprocals of each other somewhere in our proof because of the right angle at C. Not having any of the vertices at the origin or on an axis complicates our calculations as well.

Example 2:
Prove that the median drawn to the base of an isosceles triangle is perpendicular to the base.



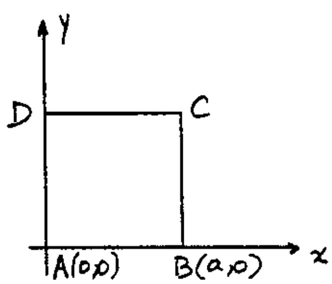


Remark: It may have been tempting to choose the coordinates of the isosceles triangle as shown at left, but this immediately would assume that the median is perpendicular to the base since the axes are perpendicular. With this orientation, one would not use the given fact that the triangle is isosceles. If we were given that the median drawn to one side of a triangle was perpendicular to that side and were then asked to prove that the triangle was isosceles, the layout would be ideal.

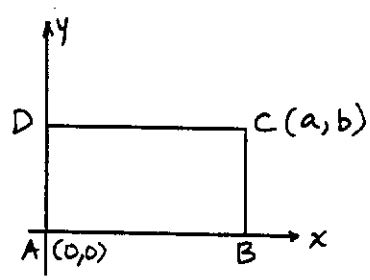
Assignment

For questions 1 to 6 determine the coordinates of the remaining points in each figure if the figure is to be:

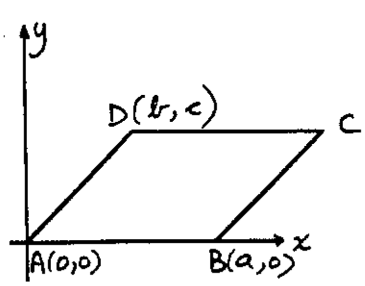
1. a square



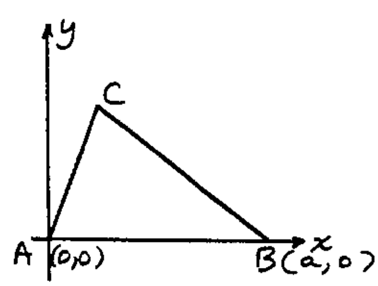
2. a rectangle



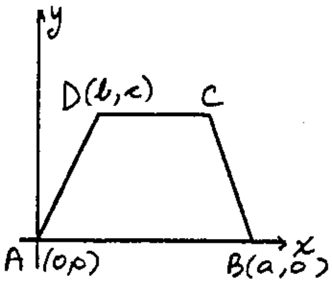
3. a parallelogram



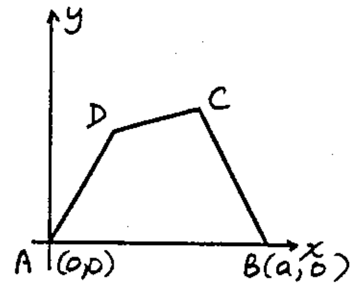
4. any triangle



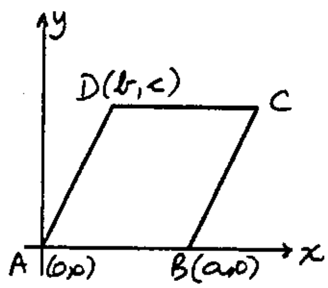
5. an isosceles trapezoid



6. any quadrilateral



7. a rhombus



Use coordinate geometry to prove each of the following statements. You can use the coordinates established in questions 1 to 7 to assist your diagram layout.

8. The diagonals of a rectangle are congruent.

9. The line segment joining the midpoints of two sides of a triangle is (a) parallel to the remaining side of the triangle and (b) is half as long as the remaining side.

10. The diagonals of a rhombus are perpendicular.

11. If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.

12. The diagonals of an isosceles trapezoid are congruent.

13. If the midpoint of one side of a triangle is equidistant from all three vertices, then the triangle is a right triangle.

14. The median of a trapezoid is parallel to the two bases and has a length equal to the average of the lengths of the bases.

15. The figure formed by joining the midpoints of consecutive sides of any quadrilateral is a parallelogram.

Lesson 8: Proofs By Mathematical Induction (CCT, IL, NUM)

What is the sum of the first n odd natural numbers? That is, what the sum of $1 + 3 + 5 + 7 + 9 + \dots + 2n - 1$ (nth odd natural number)?

To explore the answer to this question, let's gather some data by examining the sum of the first one, two, three, etc., odd natural numbers. Complete the following chart.

n, the number of odd natural numbers	Indicated Sum	S_n, the sum of the n odd natural numbers
1	1	$S_1 = 1$
2	1 + 3	$S_2 = 4$
3	1+ 3 + 5	$S_3 = 9$
4	1+ 3 + 5 + 7	$S_4 = 16$
5	1+ 3 + 5 + 7 + 9	$S_5 = 25$
6	1+ 3 + 5 + 7 + 9 + 11	$S_6 = 36$
. .		. .
n	$1 + 3 + 5 + 7 + \dots + (2n - 1)$	$S_n = ?$

The data in the chart suggests (but does not prove) that the sum of the first n odd natural numbers is n^2. How can we prove this to be true? The techniques we have learned so far are sufficient to prove simple statements such as "for all natural numbers n, $(n + 1)^2 - 2(n + 1) + 1 = n^2$ " because we merely need to simplify the left side in order to obtain the right side. How, though, will we show that $1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$?

Mathematicians prove that statements hold true for the set of all natural numbers by using a proof process known as **mathematical induction.** The process involves essentially two main steps.

Step One:
We must show that the statement to be proven holds true for the first natural number for which it claims to be true. (Often we show that the statement is true for two or three of the first natural numbers.) This step is called a **basis step** because there has to be some basis on which to claim the statement we are trying to prove is true.

Step Two:
We then assume that the statement holds for any natural number, say k . This assumption is called the **induction hypothesis**. If on the basis of this assumption, we can prove that the statement is also true for the next natural number, k+1, (this is called the **induction step)**, then the statement is considered to be true for all natural numbers.

An analogy or two may help to understand the nature of proof by mathematical induction.

Consider an infinitely long ladder reaching into the sky. What is required to climb to any height on that ladder? Essentially only two things. I need initially to be able to reach the first rung. I also need to demonstrate that having reached any particular rung, I am capable of reaching the next higher rung. If these two things are in place, I can climb as high as I desire.

Another commonly used analogy to explain the process of mathematical induction concerns an infinitely long chain of standing dominoes. What is required in order for all of the dominoes in the chain to fall down one by one? First, we must be able to knock over the initial domino. Second, we must be able to demonstrate that any domino that falls over will cause the one next in line to also fall over. If these two things are in place, then all the dominoes will fall.

Let us return to our original problem. How can we prove that
$S_n = 1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$?

Setp 1: (Basic Step)

Is the statement true for n = 1? Certainly our table on the last page shows us that the statement is true not only for n = 1, but also when n = 2, 3, 4, 5, and 6. There is definitely a basis on which to think the statement is true for all natural numbers.

Step 2:

Assume that the statement is true for any natural number k. That is, assume that $S_k = 1 + 3 + 5 + 7 + ... + (2k - 1) = k^2$. We have formed the induction hypothesis assuming that the sum of the first k odd natural numbers is k^2.

Having assumed the statement is true when n = k (it is true for the first k odd natural numbers), we must prove that the statement is true when n = k + 1, (we must prove that it is true for the first k + 1 odd natural numbers).

We must show that:

this simplifies to become 2k+1 which is the next odd natural number after 2k-1. (They differ by 2.)

$$S_{k+1} = 1 + 3 + 5 + 7 + ... + \underbrace{(2k - 1)}_{\text{the kth odd natural number}} + \underbrace{(2(k + 1) - 1)}_{\text{the k+1st odd natural number}} = (k + 1)^2 \qquad (1)$$

Can we show that the left side of statement (1) above is the same as the right side?

By our induction hypothesis $1 + 3 + 5 + 7 + ... + (2k - 1) = k^2$

$\therefore \underbrace{1 + 3 + 5 + 7 + ... + (2k - 1)} + (2(k + 1) - 1)$ (the left side of statement (1))

$= k^2 + (2(k + 1) - 1)$
$= k^2 + 2k + 2 - 1$
$= k^2 + 2k + 1$
$= (k + 1)(k + 1)$
$= (k + 1)^2$ (the right side of statement 1)

Since the left side of statement (1) is the same as the right side, the statement has been proven true to n = k + 1 (the first k + 1 odd natural numbers). Thus by the principle of mathematical induction, the statement is true for all natural numbers.

Example 3:
Prove that for all natual numbers n,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Example 4:
Prove that the number of interior diagonals in a convex polygon of n sides is $\frac{n(n-3)}{2}$, where $n \geq 3$

Example 5:
Prove that for every natural number n, $n < 2^n$?

Assignment

Use mathematical induction to prove each of the following statements for all natural numbers n.

1.
$$1+2+3+4+\ldots+n=\frac{n(n+1)}{2}$$

2.
$$2+6+10+14+\ldots+4n-2=2n^2$$

3.
$$1+2+4+8+\ldots+2^{n-1}=2^n-1$$

4.
$$1^2+2^2+3^2+4^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

5.
$$1^2+3^2+5^2+7^2+\ldots+(2n-1)^2=\frac{n(2n+1)(2n-1)}{3}$$

6. $n^2 + n$ is divisible by 2.

7. $n^3 + 2n$ is divisible by 3.

8. $n(n^2+5)$ is divisible by 6.

9. $n < n + 1$

10. $2 \leq 2^n$

11. $2^n > n^2$ for $n \geq 5$.

12. $2n \leq 2^n$

13. $x^n - y^n$ is divisible by $x - y$.

14.

$$a+(a+b)+(a+2d)+(a+3d)+\ldots+(a+(n-1)d)=\frac{n}{2}[2a+(n-1)d]$$

15. The sum of the interior angles of a convex polygon of n sides is (n-2)180.

16. n distict lines in a plane passing through a given point divide the plane into 2n regions.

17. Any postage greater than or equal to two cents can be made up exactly using only two and three cent stamps.

18. $\cos n\pi=(-1)^n$ (Hint: use the identity for cos(A+B))

Appendix H

Lesson 1: The Need For Proof (CCT, NUM)

Problem 1: A motorist drove the 300 km from Saskatoon to Meadow Lake at a speed of 100 km/h. Poor visibility caused the motorist to make the return trip at 80 km/h. What was the motorist's average speed for the trip?

(a) What is your intuitive answer to the question? 90 Km/h (a frequent guess)

(b) Let's check it out:

How long does it take to drive there? $time = \frac{dist.}{rate} = \frac{300\text{ Km}}{100\text{ Km/h}} = 3\text{ hours}$

How long does it take to drive back? $time = \frac{dist}{rate} = \frac{300\text{ Km}}{80\text{ Km/h}} = 3.75\text{ hours}$

What is the total time spent driving? $3h + 3.75h$ $= 6.75h$

What is the total distance travelled? $300\text{Km} + 300\text{Km}$ $= 600\text{ Km}$

What then is the average speed? $Av\ speed = \frac{dist.}{time} = \frac{600\text{ Km}}{6.75h} = 88.\overline{8}\text{ Km/h}$

If the total trip takes 6.75 hours, and if the average speed had been 90 km/h, you would have covered a distance of 607.5 km. It is clear that the intuitive answer of 90 km/h is too large.

Problem 2: Two containers, one holding a litre (1000 mL) of cola, the other holding a litre of coffee, are standing beside one another. A cup (250 mL) of cola is transferred to the coffee container and thoroughly mixed in with the coffee. A cup of the coffee-cola mix is then transferred back to the cola container. Is there more coffee in the cola container or is there more cola in the coffee container?

(a) What is your intuitive answer?

(b) Let's check it out.

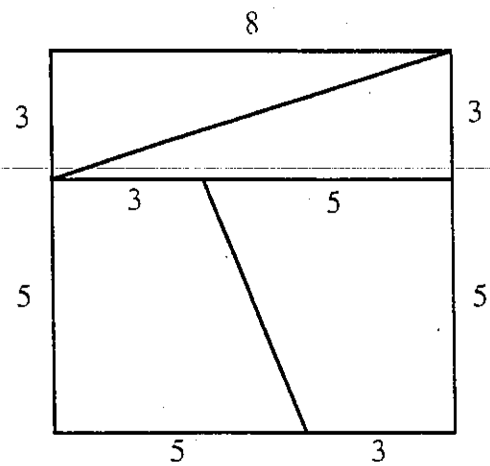
	Cola Container		Coffee Container	
Action Taken	Amount Cola	Amount Coffee	Amount Cola	Amount Coffee
original situation	1000	0	0	1000
1 cup from cola container to coffee container	750	0	250	1000
1 cup from coffee container to cola container	800	200	200	800

Cola : Coffee = 1:4
(after transfer of 1 cup from cola → coffee)
∴ $\frac{1}{5}$ of the 250 mL returned to cola container will be cola.

As you may have realized from the above examples, intuition needs to be tested by investigating situations in a precise manner.

Consider the problem below. Your intuition will tell you something is amiss. See if you can determine what that is.

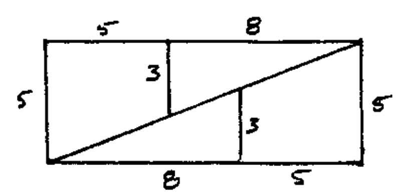
Problem 3: On a sheet of graph paper draw the figure shown below.



Use large squared paper to make pieces easy to handle.

(a) What is the area of the square above? $64u^2$

(b) From your sheet of graph paper, cut out the four pieces shown in the figure and rearrange them to form a rectangle. (You do not have to flip over any of the four pieces.)



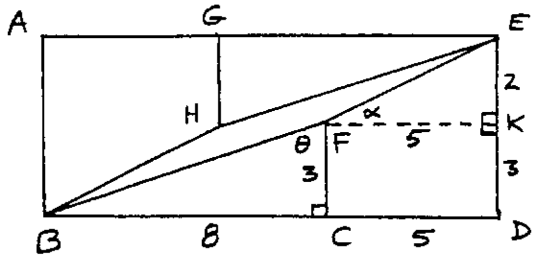
(c) What is the area of the rectangle formed in part (b)? $65u^2$

(d) What is the resulting contradiction?

It appears that rearranging the pieces causes the area to increase by $1u^2$.

(e) Explain why this contradiction arises?

The problem lies in that rectangle ABDE formed by the four pieces has "hidden" within it a parallelogram of area 1 u². The figure below has been exagerated to show the parallelogram but the calculations verify the claim above. The eye may not be able to see that points B, F, and E are not collinear but $\theta + \angle CFK + \alpha \neq 180^\circ$ (see below).



$\tan\theta = \frac{8}{3}$

$\therefore \theta = \tan^{-1}(8/3) = 69.4440^\circ$

$\angle CFK = 90^\circ$

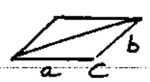
$\tan\alpha = \frac{2}{5}$, $\therefore \alpha = \tan^{-1}\frac{2}{5} = 21.8014^\circ$

$\therefore \theta + \angle CFK + \alpha = 69.4440^\circ + 90^\circ + 21.8014^\circ = 181.2454^\circ$

$\therefore \angle BFE = 360^\circ - 181.2454^\circ = 178.7546^\circ$

By Pythagorus, $BF = \sqrt{8^2+3^2} = \sqrt{73}$. Similarly, $FE = \sqrt{5^2+2^2} = \sqrt{29}$. Similar calculations show $EH = \sqrt{73}$ and $BH = \sqrt{29}$. ∴ BFEH is a llgm (opp. sides are ≅) We know that the area of a Δ is given by $\frac{1}{2}ab\sin C$, so the the area of a llgm (2 ≅ Δs) is $ab\sin C$. (see figure below)

$\therefore$ area llgm BFEH $= (\sqrt{73})(\sqrt{29})\sin 178.7546^\circ$



$= 1.0005$. (The reason this value is not exactly one is because of the rounding in calculating θ and α.

Lesson 2: Inductive Reasoning (CCT, IL)

Example 2:

(a) Complete each of the following calculations:
1 x 8 + 1 = ***9***
12 x 8 + 2 = ***98***
123 x 8 + 3 = ***987***
1234 x 8 + 4 = ***9876***

(b) Based on your calculations above, predict the answers to the following calculations.
12345 x 8 + 5 = ***98765***
123456 x 8 + 6 = ***987654***
123456789 x 8 + 9 = ***987654321***

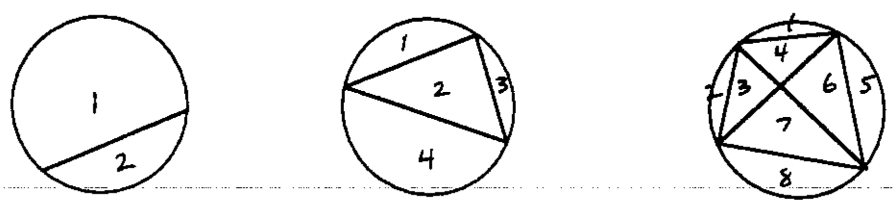
(c) Use a calculator to check your predictions in part (b).

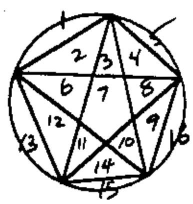
(d) Does this pattern continue? Does 12345678910 · 8 + 9 = 9876543210 ?

NO .

Example 3:

Consider the pattern suggested by the following figures.



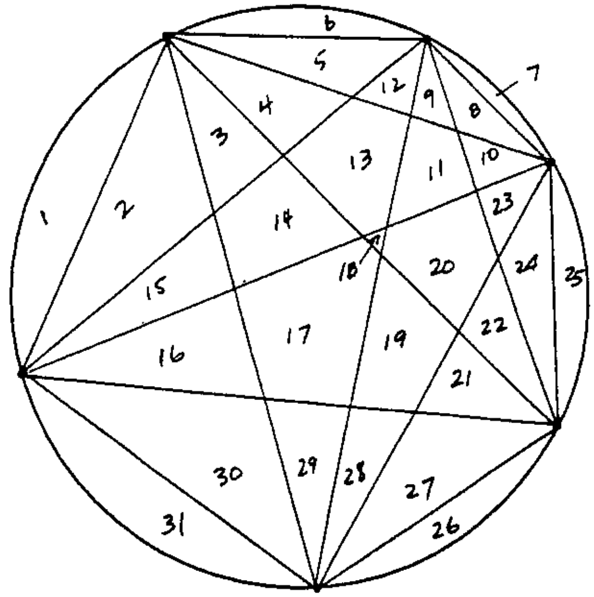


Number of points connected	2	3	4	5	6	
Number of resulting regions	2	4	8	16		

(a) Count the number of regions resulting when 3, 4, and 5 points are connected and complete the appropriate portion of the table.

(b) Based on the data collected, predict how many regions can be formed when 6 points are connected. 32

(c) Place 6 points on the circumference of the circle at the top of the next page and connect them in every possible way. Count the number of resulting regions.



only 31 regions (max) will result.

(d) What does this example illustrate about inductive reasoning?

Inductive reasoning can lead to wrong conclusions as well as correct ones.

Lesson 3: Deductive Reasoning (CCT, NUM, IL)

Example 1:
Try the following number trick. Complete the chart using three different starting numbers. The first has been done for you.

Directions	1st #	2nd #	3rd #
Choose any number	11	200	-7
Multiply by 4	44	800	-28
Add 10	54	810	-18
Divide by 2	27	405	-9
Subtract 5	22	400	-14
Divide by 2	11	200	-7
Add 3	14	203	-4
Subtract the original number	3	3	3

Inductive reasoning would suggest to us that the result would always be 3. We can use deductive reasoning to prove what inductive reasoning suggests. If we let our starting number be x, then the step by step results are shown below. Complete the chart.

If we accept the polynomial manipulations to the right of each statement as being correct, then the conclusion that we reach, namely that 3 is always the result, is also true. Once again, reasoning in this manner is termed deductive reasoning.

Directions	1st #
Choose any number	x
Multiply by 4	$4x$
Add 10	$4x + 10$
Divide by 2	$2x + 5$
Subtract 5	$2x$
Divide by 2	x
Add 3	$x + 3$
Subtract the original number	3

Example 2:
(a) Choose any two positive integers.
(b) Find the square of the sum of the integers chosen in (a).
(c) Find the sum of the squares of the integers chosen in (a).
(d) How does your answer to (b) compare in size to your answer in (d)?
(e) Use deductive reasoning to prove that your conclusion in (d) will hold for any two positive integers.

(a) 8, 9

(b) $(8+9)^2 = 17^2 = 289$

(c) $8^2 + 9^2 = 64 + 81 = 145$

(d) Square of sum > sum of squares

(e) Let a and b be the two positive integers

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$= a^2 + b^2 + 2ab$$

Since a and b are positive so is 2ab

$\therefore$ $a^2 + b^2 + 2ab$ will be $> a^2 + b^2$

$\therefore$ $(a+b)^2 > a^2 + b^2$.

Example 3:
For any two positive numbers x and y, prove that if x > y, then 1/x < 1/y.

If $\frac{1}{x} - \frac{1}{y} < 0$, then $\frac{1}{x}$ will be $< \frac{1}{y}$.

$\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy}$. Since x and y are positive,

xy is positive. Since $x > y$ then $y - x$ is negative.

$\therefore$ $\frac{y-x}{xy} = \frac{\text{a negative \#}}{\text{a positive \#}} =$ a negative #

Since $\frac{1}{x} - \frac{1}{y}$ is negative, $\frac{1}{x} < \frac{1}{y}$.

Lesson 4: Conditional Statements and Proofs By Counterexample (CCT, IL)

Example 1:
Identify the hypothesis and conclusion in the statement "If the Blades win five in a row, then I'll buy a season's ticket."

H: The Blades win five in a row
C: I'll buy a season's ticket

If-then statements are sometimes called **conditional statements** or just **conditionals**. Conditionals may not always appear in the if p then q form, but they can always be rewritten in that form.

Example 2:
Rewrite each of the following statements in the if p then q form.

(a) All parallelograms have opposite sides that are congruent.

If a figure is a parallelogram, then its opposite sides will be congruent.

(b) The square of an even integer is even.

If an integer is even, then its square will be even.

(c) Eating candy will cause your teeth to decay.

If you eat candy, then your teeth will decay.

A conditional statement can be proven to be false if an example can be found for which the hypothesis is true but the conclusion is false. Such an example is called a **counter example.** It only takes one counter example to prove that a conditional statement is false.

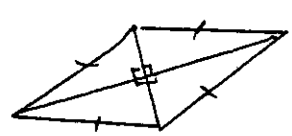
Example 3:
Provide a counter example to prove that each statement below is false.

(a) If x > 0, then √x < X.

$\frac{1}{4} > 0$ but $\sqrt{\frac{1}{4}}$ or $\frac{1}{2}$ is not less than $\frac{1}{4}$.
(The statement is false for any x value such that $0 \leq x < 1$.)

(b) If the diagonals of a quadrilateral are perpendicular, then the quadrilateral is a square.

In a rhombus the diagonals are ⊥ but a rhombus is not a square.

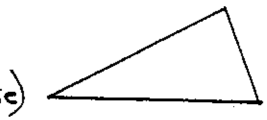


When you interchange the conclusion and the hypothesis of an if-then statement, you form the **converse** of the statement.

Original statement: if p, then q
Converse statement: if q, then p

Example 4:
Write the converse of each of the following if-then statements. Determine if the converse is true or false.

(a) If a triangle is equilateral, then it is acute.

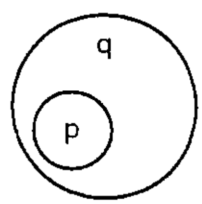
If a triangle is acute, then it is equilateral. (False) 

(b) If a quadrilateral has four right angles, then it is a rectangle.

If a quadrilateral is a rectangle, then it has four right angles. (true)

Example 4 shows us that the converse of a true if-then statement may be true or it may be false. By using Venn diagrams we can determine whether conclusions reached in an argument are valid. The statement "if p then q" can be illustrated by the following diagram.

Consider the true statement "If a figure is a square, then it is a polygon". The small circle p in the diagram below represents all figures that are squares. The larger circle, labelled q, represents all figures that are polygons. Clearly the relationship of the two circles is correct because all square figures are also polygons, so the little circle belongs in the big one.



Example 5:
Examine each of the following statements in light of the Venn diagram and determine the conclusion, if any, that can be reached.

(a) If a figure is a square, then it is a polygon. The figure I am looking at is a square.

∴ the figure is a polygon. (If you are in circle p, you are in circle q)

(b) If a figure is a square, then it is a polygon. I am looking at a polygon.

No conclusion possible. (being inside circle q doesn't mean you are inside circle p)

(c) If a figure is a square, then it is a polygon. I am not looking at a square.

No conclusion possible. (being outside circle p doesn't mean you are in circle q - you might be outside q also.)

(d) If a figure is a square, then it is a polygon. I am not looking at a polygon.

∴ I am not looking at a square. (being outside circle q means you are definitely outside circle p).

Lesson 5: Integer Property Proofs (CCT, NUM)

Example 1: Prove that the product of any two even integers is even.

Let the two even integers be a and b.
Then $a = 2k$ and $b = 2j$ where k and j are integers

$\therefore ab = (2k)(2j) = 4kj = 2(2kj)$

$\therefore ab$ is a multiple of 2 ($2 \cdot kj$ is an integer if k and j are integers — integers are closed under multiplication)

$\therefore ab$ is even.

Example 2: Prove that the square of any odd integer is odd.

Let a be any odd integer

$\therefore a = 2w + 1$ where w is an integer

$\therefore a^2 = (2w+1)^2 = 4w^2 + 4w + 1$

$= 2(2w^2 + 2w) + 1$

Since w is an integer, so is $2w^2 + 2w$.

$\therefore a^2 = 2(\text{integer}) + 1$ which means it's odd.

Example 3: Prove that the sum of a two digit number and the number formed by reversing its digits will always be divisible by 11.

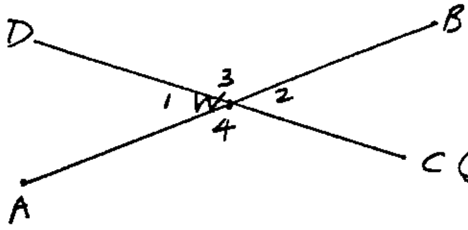
Let $10t + u$ be the original two digit number.
Then $10u + t$ is the number with its digits reversed

$\therefore 10t + u + 10u + t = 11t + 11u = 11(t + u)$ which is a multiple of 11.

Lesson 6: Deductive Geometric Proofs and Indirect Proofs (CCT, IL)

The following four examples illustrate two column deductive geometric proofs similar to the kind of proofs in the Mathematics 20 course.

Example 1: Prove that the pairs of vertical angles formed by two intersecting lines are congruent.

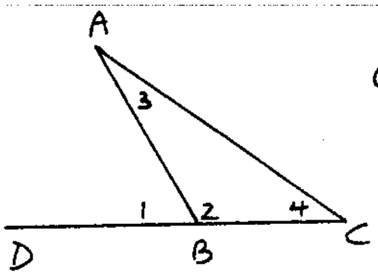


Given: $\angle 1$ and $\angle 2$ are vertical $\angle$'s
$\angle 3$ and $\angle 4$ are vertical $\angle$'s

Prove: $\angle 1 = \angle 2$; $\angle 3 = \angle 4$

	Statement	Reason
①	$\angle 1 + \angle 3 = 180°$	$\angle AWB$ is a straight $\angle$
②	$\angle 3 + \angle 2 = 180°$	$\angle DWC$ is a straight $\angle$
③	$\therefore \angle 1 + \angle 3 = \angle 3 + \angle 2$	Substitution (② into ①)
④	$\therefore \angle 1 = \angle 2$	Subtraction of $\angle 3$ in step ③.
⑤	$\angle 1 + \angle 4 = 180°$	$\angle DWC$ is a straight $\angle$
⑥	$\therefore \angle 1 + \angle 3 = \angle 1 + \angle 4$	Substitution (⑤ into ①)
⑦	$\therefore \angle 3 = \angle 4$	Subtraction of $\angle 1$ in ⑥

Example 2: Prove that the measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.



Given: $\angle 1$ is an exterior $\angle$ of $\triangle ABC$

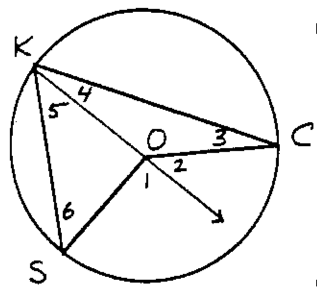
Prove: $\angle 1 = \angle 3 + \angle 4$

	Statement	Reason
①	$\angle 2 + \angle 4 + \angle 3 = 180°$	Sum of measures of $\angle$'s in a $\triangle$ = 180°
②	$\angle 1 + \angle 2 = 180°$	$\angle DBC$ is a straight. $\angle$
③	$\therefore \angle 1 + \angle 2 = \angle 2 + \angle 4 + \angle 3$	Substitution of ① into ②
④	$\angle 1 = \angle 4 + \angle 3$	subtraction of $\angle 2$ in step ③.

Example 3: Prove that the angle at the circumference of a circle is equal to one-half the angle at the center of the circle subtended by the same arc.

Given: figure as shown

Prove: $\angle CKS = \frac{1}{2}\angle COS$



Introduce $\overline{KO}$

① $\angle 2 = \angle 3 + \angle 4$ } (ext. ∠ of Δ = Σ of 2 rem. int. ∠'s.)

② $\angle 1 = \angle 5 + \angle 6$

③ $\angle 1 + \angle 2 = \angle 3 + \angle 4 + \angle 5 + \angle 6$ (addition of ① + ②)

④ $OC = OK = OS$ (= radii)

⑤ ∴ ΔKOC and ΔKOS are isos. (def'n isos. Δ)

⑥ $\angle 3 = \angle 4$; $\angle 5 = \angle 6$ (base ∠'s of isos Δ)

⑦ ∴ $\angle 1 + \angle 2 = \angle 4 + \angle 4 + \angle 5 + \angle 5$ (subst'n of ⑥ into ③)

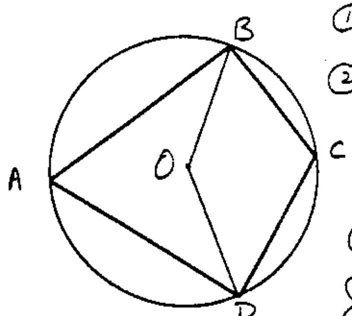
⑧ $\angle 1 + \angle 2 = 2\angle 4 + 2\angle 5$ (simplify ⑦)

⑨ $\angle 1 + \angle 2 = 2(\angle 4 + \angle 5)$ (factor ⑧)

⑩ $\angle COS = 2\angle CKS$ (rename ∠'s in ⑨)

⑪ $\frac{1}{2}\angle COS = \angle CKS$ (÷n by 2 in ⑩)

Example 4: Prove that the opposite angles of a quadrilateral inscribed in a circle (a cyclic quadrilateral) are supplementary.



① $\angle BAD = \frac{1}{2}\angle BOD$ } ∠ at circumference is half angle at center.

② $\angle BCD = \frac{1}{2}$ reflex $\angle BOD$

③ $\angle BAD + \angle BCD = \frac{1}{2}\angle BOD + \frac{1}{2}$ reflex $\angle BOD$ (+n of ① + ②)

④ $\angle BAD + \angle BCD = \frac{1}{2}(\angle BOD +$ reflex $\angle BOD)$ (factor)

⑤ $\angle BAD + \angle BCD = \frac{1}{2}(360^\circ)$ ($\angle BOD$ + reflex $\angle BOD = 360^\circ$)

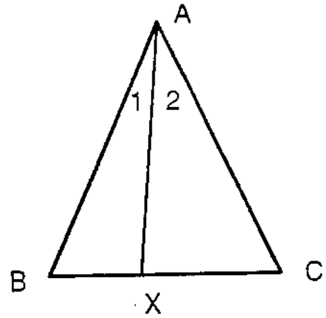
⑥ $\angle BAD + \angle BCD = 180^\circ$ (simplify ⑤)

⑦ ∴ ∠BAD and ∠BCD are supplementary (def'n of supp. ∠'s)

⑧ Similarly by introducing $\overline{AO}$ and $\overline{CO}$ we can prove ∠ABC supplements ∠ADC

Example 5:
Given:

$\angle 1 \neq \angle 2$, $\overline{AB} \cong \overline{AC}$



Prove:

$\overline{BX} \neq \overline{CX}$

① Assume $\overline{BX} \cong \overline{CX}$
In $\triangle ABX$ and $\triangle ACX$

② $\overline{AB} \cong \overline{AC}$ (given)

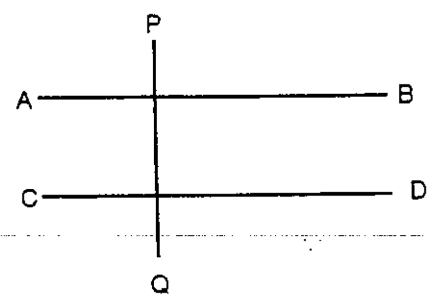
③ $\overline{AX} \cong \overline{AX}$ (common side)

④ $\therefore \triangle ABX \cong \triangle ACX$ (S.S.S. steps ①,②,③)

⑤ $\therefore \angle 1 \cong \angle 2$ (corr. parts of $\cong \triangle$'s are $\cong$)

⑥ This contradicts the given information that $\angle 1 \ncong \angle 2$. $\therefore$ our assumption in ① is false.
$\therefore \overline{BX} \ncong \overline{CX}$

Example 6:

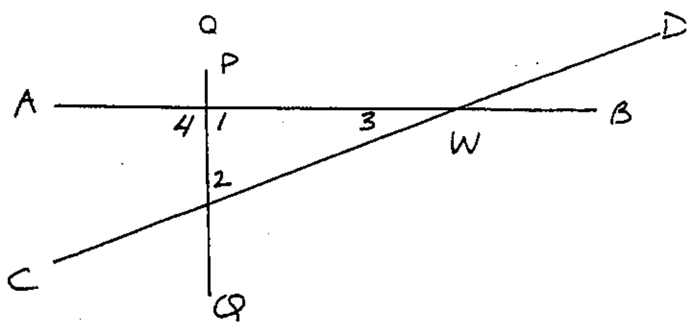


Given:

$\overline{AB} \perp \overline{PQ}$, $\overline{CD} \perp \overline{PQ}$

Prove:

$\overline{AB} \parallel \overline{CD}$



① Assume $\overleftrightarrow{AB}$ is not $\parallel$ $\overleftrightarrow{CD}$

② $\therefore$ $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect, say as in the diagram to the right

③ $\overline{CD} \perp \overline{PQ}$ (given)

④ $\therefore \angle 2$ is a right $\angle$ (defn $\perp$)

⑤ $\therefore \angle 2 = 90°$ (defn right $\angle$)

⑥ $\angle 4 = \angle 2 + \angle 3$ (ext. $\angle$ = sum of 2 remote interior $\angle$'s)

⑦ $\angle 4 = 90° + \angle 3$ (subst'n ⑤ into ⑥)

⑧ $\therefore \angle 4 > 90°$ ($90° + \angle 3 > 90°$, unless $\angle 3 = 0°$ which I am assuming it's not — that would mean $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ coincide).

⑨ $\therefore \angle 4$ is not a right $\angle$ (defn. right $\angle$)

⑩ $\therefore$ $\overline{AB} \not\perp \overline{PQ}$ (defn. $\perp$)

⑪ This contradicts the given fact that $\overline{AB} \perp \overline{PQ}$. $\therefore$ our assumption is false. $\therefore$ $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Example 7: Prove that $\sqrt{2}$ is an irrational number.

(1) Assume that $\sqrt{2}$ is rational
(2) $\therefore \sqrt{2} = \frac{a}{b}$ where a and b are integers with no common factor.
(3) $\therefore a = \sqrt{2}\, b$
(4) $\therefore a^2 = 2b^2$
(5) $\therefore a^2$ is even ((4) shows a^2 is a multiple of 2)
(6) $\therefore$ a is even (if a were odd, a^2 would be odd)
(7) $\therefore a = 2w$ (definition of even)
(8) $\therefore a^2 = 4w^2$ (squaring (7))
(9) $\therefore 4w^2 = 2b^2$ (substitute (8) into (4))
(10) $\therefore 2w^2 = b^2$ (÷ (9) by 2)
(11) $\therefore b^2$ is even (def'n even) it's a multiple of 2
(12) $\therefore$ b is even (if b were odd its square would be odd)
(13) $\therefore$ both a and b are even (Steps (6), (12))
(X) (14) $\therefore$ a and b have a common factor of 2. This contradicts the fact that a and b are integers with no common factor (step 2)
(15) $\therefore$ our assumption is false so $\sqrt{2}$ is irrational.

Example 8: To prevent Al, Betty, and Cory from dominating the High Rollers Social Club, the membership of the executive committee of that club was restricted by the conditions below. Prove that Betty is not a member of the executive.

(a) If Al is a member, then Betty is not a member.
(b) If Al is not a member, then Cory is a member.
(c) If Betty is a member, then Cory is not a member.

(p) q

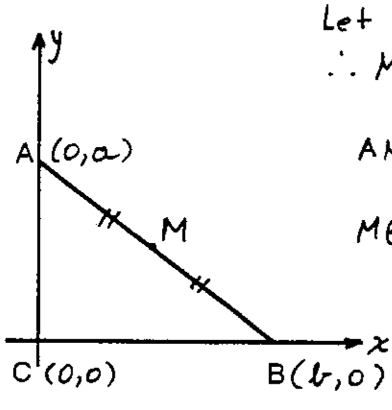
Assume Betty is a member
$\therefore$ Al is not a member (If p, then q, and not q, implies not p) statement (a)
~~$\therefore$ Cory is a member (statement (b))~~
$\therefore$ Betty is not a member (Statement (c) If p, then q, and not q, implies not p).

Example 9: Prove that there are infinitely many prime numbers.

(1) Suppose the number of prime numbers is finite
(2) Then there must be some greatest prime number, say p.
(3) Let q be the integer which is 1 greater than the product of all the prime numbers.
$\therefore q = (2)(3)(5)(7)(11)(13)\cdots(p) + 1$
(4) $\therefore$ q has no prime factors. (If we divide q by any prime we will always have a remainder of 1)
(5) $\therefore$ q is prime
(X) (6) since $q = (2)(3)(5)\cdots(p)+1$ it is greater than p.
(7) This contradicts (2). $\therefore$ our assumption is false.
$\therefore$ the number of primes is infinite.

Lesson 7: Coordinate Geometry Proofs (CCT, IL, NUM)

Example 1:
Use coordinate geometry to prove that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.



Let A and B have coordinates of (0,a) and (b,0)

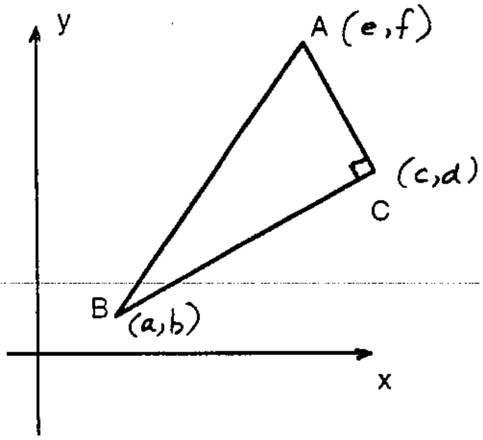
$\therefore M = \left(\frac{0+b}{2}, \frac{a+0}{2}\right) = \left(\frac{b}{2}, \frac{a}{2}\right)$

$AM = \sqrt{\left(0-\frac{b}{2}\right)^2+\left(a-\frac{a}{2}\right)^2} = \sqrt{\frac{b^2}{4}+\frac{a^2}{4}} = \frac{\sqrt{a^2+b^2}}{2}$

$MB = \sqrt{\left(b/2-b\right)^2+\left(a/2-0\right)^2} = \sqrt{\frac{b^2}{4}+\frac{a^2}{4}} = \frac{\sqrt{a^2+b^2}}{2}$

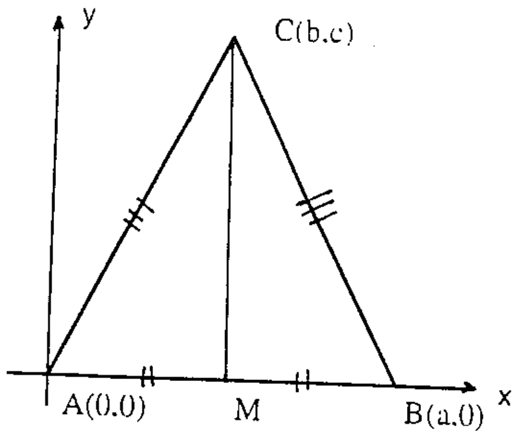
$MC = \sqrt{\left(\frac{b}{2}-0\right)^2+\left(\frac{a}{2}-0\right)^2} = \sqrt{\frac{b^2}{4}+\frac{a^2}{4}} = \frac{\sqrt{a^2+b^2}}{4}$

Since $MA = MB = MC$, M is equidistant from A, B, and C.



Remarks: The most important step in a coordinate geometry proof is placing the figure onto the coordinate plane in such a way as to make the mathematical manipulations as simple as possible. Clearly had we chosen the coordinates of the vertices of the right triangle as shown at the left, our calculations would have been much more involved. We would have had to use the fact that AC and BC have slopes that are negative reciprocals of each other somewhere in our proof because of the right angle at C. Not having any of the vertices at the origin or on an axis complicates our calculations as well.

Example 2:
Prove that the median drawn to the base of an isosceles triangle is perpendicular to the base.



Since the base, $\overline{AB}$, has a slope of 0, we must show that the slope of the median $\overline{M}$ is undefined.

$M = \left(\frac{0+a}{2}, \frac{0+0}{2}\right) = \left(\frac{a}{2}, 0\right)$

$\therefore$ slope of $\overline{CM} = \dfrac{c-0}{b-\frac{a}{2}} = \dfrac{c}{\frac{2b-a}{2}} = \dfrac{2c}{2b-a}$

because $\triangle ABC$ is isosceles, $AC = BC$

$AC = \sqrt{(0-b)^2+(0-c)^2} = \sqrt{b^2+c^2}$

$BC = \sqrt{(a-b)^2+(0-c)^2} = \sqrt{a^2-2ab+b^2+c^2}$

$\therefore \sqrt{b^2+c^2} = \sqrt{a^2-2ab+b^2+c^2}$

$\therefore b^2+c^2 = a^2-2ab+b^2+c^2$

$0 = a^2-2ab$

$0 = a(a-2b)$

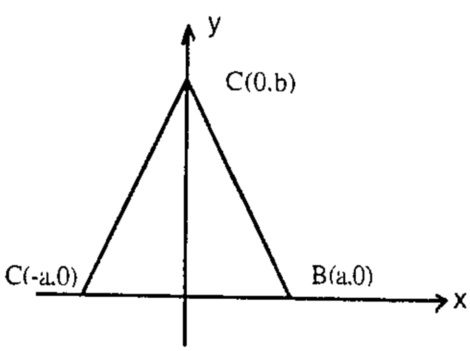
$\therefore a = 0$ (impossible) or $a-2b=0$

if $a-2b=0$, slope of $\overline{CM}$ is

$\dfrac{2c}{2b-a} = \dfrac{2c}{-1(a-2b)} = \dfrac{2c}{-1(0)} = \dfrac{2c}{0}$ = undefined.

$\therefore \overline{CM} \perp \overline{AB}$

Here's an example of a coordinate proof being longer than a 2 col. proof.



Remark: It may have been tempting to choose the coordinates of the isosceles triangle as shown at left, but this immediately would assume that the median is perpendicular to the base since the axes are perpendicular. With this orientation, one would not use the given fact that the triangle is isosceles. If we were given that the median drawn to one side of a triangle was perpendicular to that side and were then asked to prove that the triangle was isosceles, the layout would be ideal.

Example 3:
Prove that for all natual numbers n,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Basis Step: Is it true when $n=1$?
Does $\frac{1}{1\cdot 2} = \frac{1}{2}$? Yes

what if $n=3$?
Does $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4} \stackrel{?}{=} \frac{3}{4}$

$\frac{1}{2}+\frac{1}{6}+\frac{1}{12} \stackrel{?}{=} \frac{3}{4}$

$\frac{6}{12}+\frac{2}{12}+\frac{1}{12} \stackrel{?}{=} \frac{3}{4}$

$\frac{9}{12} \stackrel{?}{=} \frac{3}{4}$ (yes)

Induction hypothesis:
Assume that the statement is true for any natural number K. That is
assume that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Prove now that the statement is true when $n=k+1$.
i.e. Prove that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+1+1)}=\frac{k+1}{k+1+1}$$

$\frac{k+1}{k+2}$

induction hypothesis $\Rightarrow$ $\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)(k+1)}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}$$

Since L.H.S. = R.H.S. the statement is true when $n=k+1$. $\therefore$ the statement is true for all natural numbers.

Example 4:
Prove that the number of interior diagonals in a convex polygon of n sides is $\frac{n(n-3)}{2}$, where $n \geq 3$

Basis Step. Is the statement true when $n=3$? A convex polygon of three sides is a Δ. a Δ has 0 interior diagonals. If $n=3$, $\frac{3(3-3)}{2}=0$. ∴ it is true for $n=3$. If $n=4$, the polygon is a quad. which has 2 interior diagonals. $\frac{4(4-3)}{2}=2$.

Induction hypothesis: Assume that a convex polygon of K sides has $\frac{K(K-3)}{2}$ interior diagonals.

We must now prove that the statement is true if $n=K+1$. That is we must show that a convex polygon of $K+1$ sides has $\frac{(K+1)(K+1-3)}{2}$

or $\frac{(K+1)(K-2)}{2}$ diagonals

The extra vertex obtained when we change from a k gon to a k+1 gon can be connected to $k+1-3$ vertices to form diagonals. (To form diagonals you can't connect the k+1st vertex to itself nor to the two adjacent vertices hence the $k+1-3$). By introducing the k+1 st vertex you however will allow for 1 more diagonal formed by connecting the two vertices adjacent to (one on either side of) the K+1 st vertex. Thus the total number of new diagonals formed by adding the K+1st vertex is $K+1-3+1$ or $K-1$ diagonals.

Since a convex polygon of K sides has $\frac{K(K-3)}{2}$ interior diagonals (induction hypothesis) and since creating a K+1 gon adds $K-1$ diagonals to the figure, the total number of diagonals is $\frac{K(K-3)}{2}+K-1=\frac{K(K-3)+2(K-1)}{2}=$

$\frac{K^2-3K+2K-2}{2}=\frac{K^2-K-2}{2}=\frac{(K-2)(K+1)}{2}$. This is what we had to show to prove the statement true for $n=K+1$. ∴ true for all natural numbers ≥ 3.

Example 5:
Prove that for every natural number n, $n < 2^n$?

Basis step: Is the statement true for n = 1, 2, 3 ?
$1 < 2^1$, $2 < 2^2$, $3 < 2^3$ – yes all are true.

Induction hypothesis. Assume the statement is true for n = k
That is assume that $k < 2^k$.
We must now prove that the statement is true when n = k+1. We must show that
$k+1 < 2^{k+1}$

By our assumption $k < 2^k$
Multiplying both sides by 2 yields $2k < 2 \cdot 2^k$ ①
$\therefore k+k < 2^{k+1}$ ①

but k > 1 or 1 < k since we Know our statement is true for n = 1 our assumption that the statement is true for any number k implies k > 1
So if $1 < k$ then
$1+k < k+k$ or $1+k < 2k$
$\therefore 1+k < 2^{k+1}$ (see 1)
∴ our statement is true when n = k+1. ∴ it is true for all natural numbers.

Appendix I

Mathematical Proof - Assignment Solutions

The Need for Proof
SOLUTIONS

1. To average 100km/h, the time taken to drive the 500km round trip would be $\frac{500km}{100\frac{km}{h}}$ or 5 hours. However the time taken to get to Saskatoon is $\frac{250km}{50km/h}$ or 5 hours. ∴ the motorist would have to return in 0 hours. That's impossible. There is no speed at which the motorist could return for this to happen.

2. $C = \pi d$

$\therefore \frac{C}{\pi} = d$

when $C = 40\,000km$, $d = \frac{40\,000km}{\pi} = 12732.39545km$ or $12732395.45m$

when $C = 40\,000.002km$ $d = \frac{40\,000.002}{\pi} = 12732.39608km$ or $12732396.08m$

The difference in diameters is .63397m or 63.4cm. If the ring remains the same distance from the earth there would be $\frac{63.4}{2}$ cm or 31.7cm (more than a foot) of space to crawl under the ring. You could easily crawl under. If the ring is gathered as below then you'd have the full 63.4cm.

3. The problem lies in that the wealthy Arab did not divide up his entire estate. $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{9}{18} + \frac{6}{18} + \frac{2}{18} = \frac{17}{18}$. The Arab has not accounted for $\frac{1}{18}$th of his estate.

4. (a) If you saved 100% on fuel you wouldn't need any in order to travel — impossible

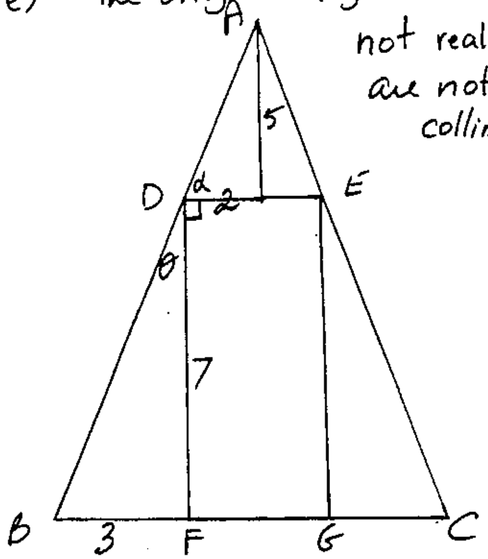
(b) If 100L were normally used then only 90L would be needed to get by the first device; 60% of the 90L would be used to get by the second device i.e. 54L and 50% of the 54L or 27L would be used to get by the third device. Thus the actual saving would be 73L of the original 100L so there'd be a 73% saving

The Need for Proof
SOLUTIONS

5 (b) base = 10 units; altitude = 12 units, area = $\frac{1}{2}(10)(12) = 60u^2$.

(d) base = 10 units; altitude = 12 units, area = $\frac{1}{2}(10)(12) = 60u^2$.
There is a hole in the center whose area is $1(2)$ or $2u^2$. This leaves an apparent area of $60-2$ or $58u^2$

(e) The original figure is not really a triangle. $\overline{AB}$ and $\overline{AC}$ are not really line segements, that is, A, D, and B are not collinear; likewise A, E, and C are not collinear.



$\tan\theta = \frac{3}{7}$

$\theta = \tan^{-1}(3/7)$

$\theta = 23.19859051°$

$\tan\alpha = \frac{5}{2}$

$\alpha = \tan^{-1}\frac{5}{2}$

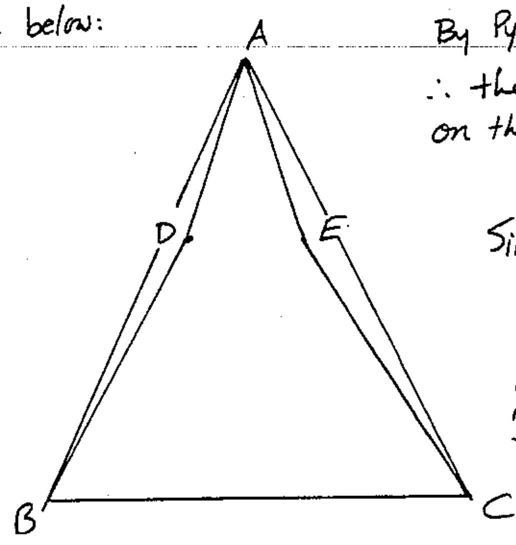
$\alpha = 68.19859051°$

$\angle FDE = 90°$

$\therefore \theta + \angle FDE + \alpha = 181.397181°$

$\therefore \angle ADB = 360° - 181.397181° = 178.602819°$

Thus there is a kink (dogsleg) as you travel from B to D to A. In exaggerated form it looks like the figure below:



By Pythagorus $AD = \sqrt{29}$ and $BD = \sqrt{58}$

$\therefore$ the area of $\triangle ADB$ (the extra piece on the left) is $\frac{1}{2}\sqrt{29}\sqrt{58}\sin 178.602819°$

$= .5$ or $\frac{1}{2}u^2$.

Similarly $\triangle AEC$ has an area of $.5u^2$

$\therefore$ the original figure has an area of $60u^2 - 2(.5u^2) = 59u^2$.

You can verify this by adding together the areas of the individual pieces.

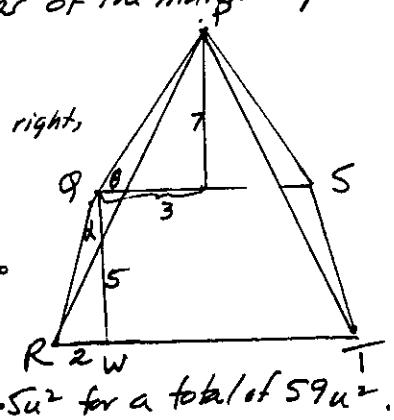


Figure 2 has a kink the other way — see at right, so the area we got, $58u^2$, is too small.

Note: $\alpha + \angle WQS + \theta = \tan^{-1}\left(\frac{2}{5}\right) + 90° + \tan^{-1}\left(7/3\right)$

$= 178.602819°$

$\therefore \triangle PQR$ has area $\frac{1}{2}(\sqrt{29})(\sqrt{58})\sin 178.602819°$

$\therefore$ area $\triangle PQR = .5u^2$. Likewise $\triangle PST$ has area $.5u^2$

$\therefore$ The correct area of the second figure is $58u^2$ plus the 2 mini $\triangle$'s each of area $.5u^2$ for a total of $59u^2$.

Inductive Reasoning
SOLUTIONS

1 a) $1 = 1^2$ ✓; $1+3 = 4 = 2^2$; $1+3+5 = 9 = 3^2$; $1+3+5+7 = 16 = 4^2$;
$1+3+5+7+9 = 25 = 5^2$

b) $1+3+5+7+9+11 = 6^2$
$1+3+5+7+9+11+13 = 7^2$

c) 10^2 or 100

d) $1+3+5+7+9+11+13+15+17+19 = 100 = 10^2$

e) yes

f) inductive reasoning is being used because we are forming a conclusion based on observations.

2 (a) 2 colors (b) 3 colors (c) 3 colors (d) 4 colors (e) 4 colors

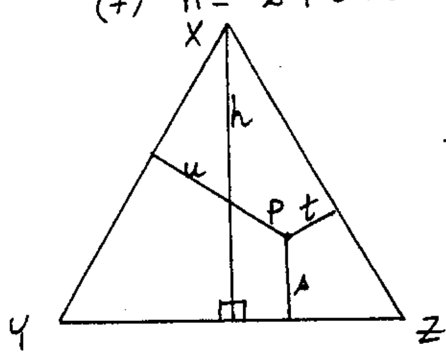
(f) that's impossible.

(g) only four colors (maximum) are needed to color a map and distinguish the borders.

(h) No. our conclusion was based only on our attempts in (f).

3 (e) answers will vary

(f) $h = s + t + u$ (see explanation below as to why.)



Area △ XYZ = Area △ PYZ + Area △ XPZ + Area △ XPY

$\frac{1}{2}(YZ)h = \frac{1}{2}(YZ)s + \frac{1}{2}(XZ)t + \frac{1}{2}(XY)u$ ①

but $YZ = XZ = XY$, so ÷ding ① by $\frac{1}{2}YZ$

we have $h = s + t + u$

Inductive Reasoning (Solutions)

4(a)
$\begin{cases} 23 \cdot 64 = 1472 \\ 32 \cdot 46 = 1472 \end{cases}$ $\begin{cases} 26 \cdot 93 = 2418 \\ 62 \cdot 39 = 2418 \end{cases}$ $\begin{cases} 41 \cdot 28 = 1148 \\ 14 \cdot 82 = 1148 \end{cases}$

$\begin{cases} 69 \cdot 64 = 4416 \\ 96 \cdot 46 = 4416 \end{cases}$ $\begin{cases} 84 \cdot 36 = 3024 \\ 48 \cdot 63 = 3024 \end{cases}$

(b) $53 \cdot 71 = 3763$, $35 \cdot 17 = 595$ $\therefore$ it is not true for all numbers of 2 digits.

(c) For each of the pairs of numbers listed the product of the ten's digits is the same as the product of the units digits. In $84 \cdot 36$ for example, $(8)(3) = (4)(6)$. This can also be proven. If the two numbers are of the form

$$\begin{cases} \underline{a}\ \underline{b} \cdot \underline{c}\ d \\ \underline{b}\ \underline{a} \cdot \underline{d}\ c \end{cases}$$

(10's digit, unit's digit)

then $(10a+b)(10c+d) = (10b+a)(10d+c)$

$$100ac + 10ad + 10bc + bd = 100bd + 10bc + 10ad + ac$$

$$\therefore 100ac + bd = 100bd + ac$$

$$\therefore 99ac - 99bd = 0$$

$$\therefore ac - bd = 0$$

$$\therefore ac = bd$$

A fun exercise is to find all two digit numbers that have that property.

5 (a)

8	9	10	11	12	13	14	15	16	17	18	19
113	131	151	173	197	223	251	281	313	347	383	421

20	21	. . .	41
461	503		(1763) — $1763 = 41 \cdot 43$

Since $f(x) = x^2 + x + 41$, if $x = 41$, then

$$41^2 + 41 + 41 = 41(41+1+1) = 41(43).$$

The program, listed on the next page, can be used on the TI-82 to find the prime factors of a number. It is pretty well essential if you want to do this problem in < 15 minutes.

Inductive Reasoning (Solutions)

```
PROGRAM: PRMFCTRS
: Input "NMBR?", N
: 2→P
: Lbl 1
: If int (N/P) = N/P
: Goto 3
: IS>(P, √N) : Goto 1      this is found under PRGM CTL.
                           It means increment and skip if
                           greater than
: Goto 4
: Lbl 3
: N/P→N
: Pause P  ←  pause + print the value of P.
: Goto 1
: Lbl 4
: If N≠1 : Disp N
```

5(b) Inductive reasoning might lead us to believe that x^2+x+41 always results in a prime number.

(c) Inductive reasoning suggests things that we may then be able to prove. It informs us of the observed patterns but does not prove that the patterns always hold.

6. 2^{20-1} or 2^{19} or 524 288

Deductive Reasoning Solutions

1. To win you must take ~~toothpick~~ #1. You can do this if you pick up ~~toothpick~~ #5 (and stop there). You'll always be able to get #5 if you get #9 (and stop there). Likewise you need to pick up 13, and 17. So to guarantee a win, let your opponent start. That way you'll be able to get #17 on your first turn. The sequence of "winning ~~toothpicks~~" is 17, 13, 9, 5, 1 whereas when you take 1 or 2 ~~toothpicks~~ the winning sequence was 19, 16, 13, 10, 7, 4, 1.

2.

(a) Choose any non zero number	x	10
(b) Double the number	$2x$	20
(c) Subtract 1	$2x-1$	19
(d) Triple your answer	$6x-3$	57
(e) Add 3	$6x$	60
(f) Divide by the original number	6	6

3. If x matches are removed there will be $20-x$ matches remaining. Since $1 \le x \le 9$, the ten's digit of $20-x$ must be a 1 and the units digit of $20-x$ will be $10-x$. $\therefore$ the sum of the digits of the number of matches left in the book will be $1+10-x$ or $11-x$. If $11-x$ matches are now removed there will only be $20-x-(11-x)$ or 9 matches left. Tearing out 2 more matches leaves 7.

Another proof (proof by exhaustion) may be easier for some students.

20	20	20	20	20	20	20	20	20
−1	−2	−3	−4	−5	−6	−7	−8	−9
19	18	17	16	15	14	13	12	11
−10	−9	−8	−7	6	5	4	3	2
9	9	9	9	9	9	9	9	9
−2	−2	−2	−2	−2	−2	−2	−2	−2
7	7	7	7	7	7	7	7	7

4. $x^2+8x+7 = (x+7)(x+1)$. Provided $x \neq 0$, in which case x^2+8x+7 would give 7, x^2+8x+7 will be factorable.

Deductive Reasoning Solutions

5(a) By pulling out only 2 socks you could have 1 red and 1 white. By pulling out the third sock you will definitely have a pair.

(b) To get 2 pairs we know that 3 socks will certainly provide 1 pair leaving 1 extra sock. The 4th sock may not match the extra sock but the 5th sock will match either the 4th sock or the extra sock. ∴ 5 socks are required

(c) To get x pairs of socks, the minimum number of socks I would need would be $2x$ socks. Now $2x$ is an even number of socks. If I had an even number of reds and an even number of whites, then $2x$ socks could provide x pairs. However this $2x$ even number of socks that are pulled out could consist of an odd number of reds and an odd number of whites since two odd numbers also add to make an even number. If I have an odd number of reds and an odd number of whites I won't be able to form x pairs. I will be able to form $x-1$ pairs having 1 extra red and 1 extra white. To guarantee my final pair of socks I will have to pull out 1 more sock for a total of $2x+1$ socks.
∴ $2x+1$ socks are required to guarantee x pairs.

6. If the person putting down the coins covers either 2 dark squares or 2 white squares the person with the paper clips can't win. With seven paper clips, 14 squares will have to be covered 7 of which will be dark and 7 of which will be white (each clip covers 1 dark and 1 white square). Thus if the person with the coin has covered say 2 dark squares, the board remains with 6 dark squares and 8 white squares making it impossible to cover 7 of each color.

7. The three were grandfather, father and son.

8.
$$b + c = 106$$
$$b - c = 100$$
$$2b = 206$$
$$b = 103$$
$$\therefore c = 3$$

many think the bottle costs \$1 and the cork costs \$.06.

Deductive Reasoning Solutions

9. It takes a cat 3 minutes to catch a mouse.
   ∴ 100 cats catch 100 mice in 3 minutes also.

10. If it were X's turn, X would win. ∴ it must be O's turn. Since there are two of each already marked, O must have started.

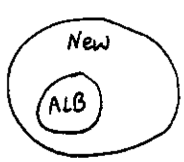
Conditional Statements and Proofs By Counter Example (Solutions)

1. hypothesis: "there's a will"
   conclusion: "there's a way"
   converse: If there's a way then there's a will.
   Whether the converse is true is argueable. Judging by the actions of some people their way through life shows little will. Many have no desire or goals. So is their will to have no will?

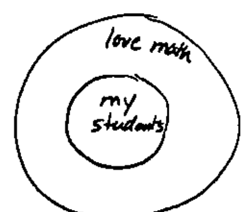
2. hypothesis: $-5x+1<16$
   Conclusion: $x>-3$
   converse: If $x>-3$ then $-5x+1<16$
   converse is true.

3. hypothesis: $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays
   Conclusion: $\angle BAC$ is a straight angle
   Converse: If $\angle BAC$ is a straight angle, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.
   Converse is true.

4. hypothesis: you are registered in a math class
   conclusion: you will need a calculator
   Converse: If you need a calculator, then you are registered in a math class.
   converse is false.

5. 

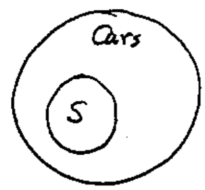
   a) no conclusion possible
   b) no conclusion possible
   c) it does not have anti lock brakes

6. 

   a) you are not my student
   b) no conclusion possible
   c) no conclusion possible

7. 

   a) I'll pick you up
   b) It did not rain
   c) no conclusion possible
   d) no conclusion possible

8. 

   a) You don't own a Saturn.
   b) no conclusion possible.
   c) no conclusion possible.

Conditional Statements and Proofs By Counter Example.
(SOLUTIONS)

9. 2 is prime but it's even

10. $(-5)^2 = 5^2$ but $-5 \neq 5$

11. $\frac{3}{2} \not< \frac{3+1}{2+1}$ since $1\frac{1}{2} > 1\frac{1}{3}$

12. If $j = -1$ and $K = 2$ then $-\frac{1}{2} \not< \frac{2(-1)}{2}$ since $-\frac{1}{2} > -1$

13 There are no primes between 90 and 95, for example, since $91 = 13 \cdot 7$, $92 = 46 \cdot 2$, $93 = 31 \cdot 3$, $94 = 47 \cdot 2$.

14. First he took the duck across the river. Then he took the fox across and returned with the duck. Then he took the corn across and returned back to get the duck.

15. The man went to town on Tuesday. He couldn't have gone on Thursday or Friday because the meat market was not open. He couldn't have gone on Saturday because the eye doctor was closed. He must have been at the bank to cash his check since he came home with more money than he left with. Thus he must have gone to town on Tuesday the only remaining day that the bank was open and the meat market and the eye doctor. (Assume that he shopped on the same day he went to the bank.)

16. Suppose the boys names were Al, Ben, and Carl. Suppose Al was the one to raise his hand. He would have reasoned as follows. "If I don't have dirt on my head then Ben or Carl raised their hands the first time because each of them saw the dirt on each other's head and didn't see any on mine. If that were true then surely Ben or Carl would realize why the other had raised his hand the first time and each would know that he had dirt on his head and would be raising his hand the second time. Because neither Ben nor Carl is raising his hand the second time it must not be that I don't have dirt on mine. Therefore I do have dirt on mine."

Conditional Statements and Proofs By Counter Example
(SOLUTIONS)

17. His friend was a lady, not a man, and her name was Anne.

18. Since their ages multiplied to 36, they must have been 1, 6, and 6 or 2, 2, and 9. In each case the sum of their ages is the same. (Any other possible combinations of ages, like 1, 4, 9 or 2, 3, 6 give unique sums.) Since the oldest girl did not like chocolate pudding their ages must have been 2, 2, and 9 because if they had been 1, 6, and 6 there could not have been an oldest girl.

Proofs About The Properties Of Integers

SOLUTIONS

1. Let the two odd integers be b and c

$\therefore b = 2k+1$ and $c = 2j+1$ where k and j are integers

$\therefore b+c = 2k+1+2j+1$

$= 2k+2j+2$

$= 2(k+j+1)$

$\therefore b+c$ is even since $2(k+j+1)$ is even.

2. Let the even integer be b

Let the odd integer be c

$\therefore b = 2k$ and $c = 2j+1$

$\therefore b+c = 2k+2j+1$

$= 2(k+j)+1$

$\therefore b+c$ is odd since $2(k+j)+1$ is odd.

3. Let the two odd integers be b and c.

$\therefore b = 2k+1$

$\therefore c = 2j+1$

$bc = (2k+1)(2j+1)$

$= 4kj+2k+2j+1$

$= 2(2kj+k+j)+1$

$\therefore bc$ is odd since $2(2kj+k+j)+1$ is the form of an odd integer

4. Let the two consecutive integers be a and $a+1$.

Now if a is even, then $a = 2k$ and $a+1 = 2k+1$

$\therefore a(a+1) = 2k(2k+1) = 4k^2+2k = 2(2k^2+k)$ which is even.

Now if a is odd, then $a = 2j+1$ and $a+1 = 2j+2$.

$\therefore a(a+1) = (2j+1)(2j+2) = 4j^2+4j+2j+2$

$= 4j^2+6j+2$

$= 2(2j^2+3j+1)$ which is even.

$\therefore$ the product of 2 consecutive integers will always be even.

Proofs About The Properties Of Integers

SOLUTIONS

5. Let the three consecutive integers be x, $x+1$, and $x+2$.

$\therefore$ their sum $= x + x + 1 + x + 2$

$= 3x + 3$

$= 3(x+1)$ which is a multiple of 3.

6. Suppose t is a common factor of x and y.

Then $x = tk$ and $y = tj$

$\therefore \; x + y = tk + tj$

$= t(k+j)$

$\therefore$ t is a common factor of their sum.

7. Let the even integer be c

$\therefore c = 2k$

$\therefore c^2 = (2k)(2k) = 4k^2 = 2(2k^2)$ which is even in form.

8. Let the two consecutive integers be x and $x+1$.

$\therefore (x+1)^2 - x^2 = x^2 + 2x + 1 - x^2 = 2x + 1$ which is in odd form.

9. Let the two odd integers be x and $x+2$

$\therefore x^2 + (x+2)^2 = x^2 + x^2 + 4x + 4 = 2x^2 + 4x + 4 = 2(x^2 + 2x + 2)$

which is of even form.

10. Let the ten's digit be t. Since the number ends in 5 it can be written as $10t + 5$.

$\therefore (10t+5)^2 = 100t^2 + 100t + 25$

$= 100t\,(t+1) + 25$

(t ↑ 10's digit; $(t+1)$ ↑ next bigger digit; ↑ attach the 25)

Proofs About The Properties Of Integers
(Solutions)

11. Let the two digit number be $10t+u$
The number with digits reversed is $10u+t$

$$10t+u-(10u+t)$$
$$=10t+u-10u-t$$
$$=9t-9u$$
$$=9(t-u)$$ which is of the form to be a multiple of 9.

12. Let x be the integer. If x is even, then $x=2k$.
$\therefore x^2=(2k)^2=4k^2$ which is a multiple of 4. If x is odd then $x=2k+1$. $\therefore x^2=(2k+1)^2=4k^2+4k+1=4(k^2+k)+1$ which is of the form to be 1 more than a multiple of 4.

Deductive Geometric Proofs and Indirect Proofs

SOLUTIONS

1. Assume that $\overline{PW} \cong \overline{QW}$. Since $\overline{PQ}$ bisects $\overline{AB}$ then $\overline{AW} \cong \overline{BW}$. Also $\angle AWP \cong \angle QWB$ since they are vertical angles. Thus $\triangle AWP \cong \triangle QWB$ by SAS. $\therefore$ $\overline{AP} \cong \overline{QB}$ (corresponding parts of $\cong \triangle$'s). This contradicts the given fact that $\overline{AP} \ncong \overline{BQ}$. $\therefore$ our assumption is false. $\therefore$ $\overline{PW} \ncong \overline{QW}$.

2. Assume that $\overline{WX} \cong \overline{ZY}$. Since $\overline{WX} \perp \overline{XY}$ and $\overline{ZY} \perp \overline{XY}$ then $\angle WXY \cong \angle ZYX$ since both are right angles. Also $\overline{XY} \cong \overline{YX}$ (common side) so $\triangle WXY \cong \triangle ZYX$ (S.A.S.) $\therefore$ $\overline{YW} \cong \overline{XZ}$ (corresponding parts of $\cong \triangle$'s). This contradicts the given fact that $\overline{YW} \ncong \overline{XZ}$. $\therefore$ our assumption is false. $\therefore$ $\overline{WX} \ncong \overline{ZY}$.

3. Assume that $\overline{SX} \cong \overline{WY}$. Now $\overline{ST} \cong \overline{WT}$ (given) so $\angle S \cong \angle W$ (In $\triangle STW$ if $\overline{ST} \cong \overline{WT}$ then the angles opposite these $\cong$ sides are also $\cong$). $\therefore$ $\triangle STX \cong \triangle WTY$ (S.A.S.) $\therefore$ $\overline{XT} \cong \overline{YT}$ (corres. parts of $\cong \triangle$'s). This contradicts the given fact that $\overline{XT} \ncong \overline{YT}$. $\therefore$ our assumption is false. $\therefore$ $\overline{SX} \ncong \overline{WY}$.

4. Assume that $\overline{SX} \cong \overline{WY}$. Now $\overline{ST} \cong \overline{WT}$ (given) so $\angle S \cong \angle W$ ($\cong \angle$'s lie opposite $\cong$ sides in a $\triangle$.) $\therefore$ $\triangle SXT \cong \triangle WYT$ (S.A.S.) so $\angle SXT \cong \angle WYT$ (corres. parts of $\cong \triangle$'s). $\therefore$ $\angle TXY \cong \angle TYX$ (they supplement $\cong \angle$'s $\angle SXT$ and $\angle WYT$ since $\angle SXY$ and $\angle WYX$ are straight angles. This contradicts the given fact that $\angle TXY \ncong \angle TYX$. $\therefore$ our assumption is false. $\therefore$ $\overline{SX} \ncong \overline{WY}$.

5. Assume that there is a largest integer k.
Since the integers are closed under addition (that is the sum of any two integers is also an integer) then $k+1$ is an integer.
Since k is the largest integer, then $k+1 < k$
$\therefore$ $1 < 0$ (Subtract k from each side)
Clearly $1 \nless 0$. $\therefore$ our assumption was false
$\therefore$ there is no largest integer

Deductive Geometric Proofs and Indirect Proofs
(SOLUTIONS)

6. Assume that $\sqrt{3}$ is rational number. Then $\sqrt{3} = \frac{a}{b}$ where a and b are integers with no common factor.

$\therefore\ a = \sqrt{3}\,b$

$\therefore\ a^2 = 3b^2$

$\therefore\ a^2$ is a multiple of 3. If a^2 is a multiple of 3, then so is "a" a multiple of 3. (Only multiples of 3 have squares that are multiples of 3. Consider $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$, etc. only $3^2, 6^2, 9^2$, etc are multiples of 3).

If "a" is a multiple of 3 then "a" = 3w where w is an integer. $\therefore$ since $a^2 = 3b^2$, then $(3w)^2 = 3b^2$ or $9w^2 = 3b^2$ so $3w^2 = b^2$. $\therefore\ b^2$ is a multiple of 3 and thus b is a multiple of 3. $\therefore$ both a and b are multiples of 3. This contradicts the fact that if $\sqrt{3}$ is rational "a" and b have no common factor. $\therefore$ our assumption is false. $\therefore\ \sqrt{3}$ is irrational.

7. Assume that the product of two even integers is odd. Then if "a" and "b" are the even integers, $ab = 2k+1$ where k is an integer. Since "a" and "b" are even integers then $a = 2w$ and $b = 2j$ where w and j are integers

$\therefore\ ab = (2w)(2j) = 2k+1$

$\therefore\ 4wj = 2k+1$

$\therefore\ 4wj - 1 = 2k$

$\frac{4wj-1}{2} = k$

$2wj - \frac{1}{2} = k$

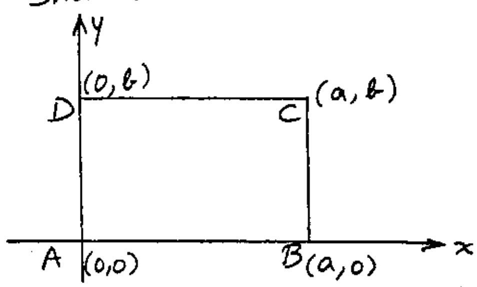
Since 2, w, and j are integers, so is 2wj. But $2wj - \frac{1}{2}$ could not be an integer. This contradicts the fact that k is an integer.

$\therefore$ our assumption is false

$\therefore$ the product of two even integers is even.

Coordinate Geometry Proofs
(SOLUTIONS)

1. C(a,a); D(0,a)
2. B(a,0); D(0,b)
3. C(a+b, c)
4. C(b,c) or any two other variables that are not the same as "a" or each other.
5. C(a−b, c)
6. C(b,c); D(d,e)
7. C(a+b, c)
8. Let the coordinates of the vertices of the rectangle be as shown:

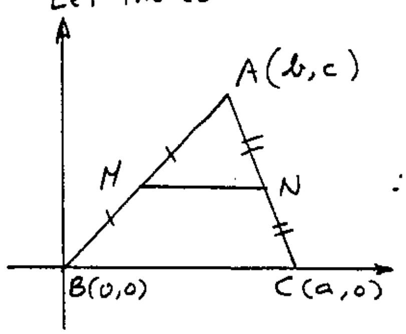


$AC = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2+b^2}$

$BD = \sqrt{(0-a)^2 + (b-0)^2} = \sqrt{a^2+b^2}$

$\therefore AC = BD.$

9. Let the coordinates of the Δ be as shown.



M's coordinates are: $\left(\frac{b+0}{2}, \frac{c+0}{2}\right) = \left(\frac{b}{2}, \frac{c}{2}\right)$

N's coordinates are: $\left(\frac{b+a}{2}, \frac{c+0}{2}\right) = \left(\frac{b+a}{2}, \frac{c}{2}\right)$

$\therefore \overline{MN}$ has a slope of $\dfrac{c/2 - c/2}{\frac{b+a}{2} - \frac{b}{2}} = \dfrac{0}{a/2} = 0.$

$\overline{BC}$ also has a slope of 0 (x axis)

$\therefore \overline{MN} \parallel \overline{BC}.$

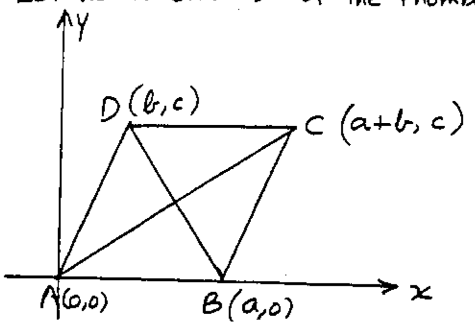
The length of $\overline{BC} = a$. The length of $\overline{MN} = \sqrt{\left(\frac{b+a}{2} - \frac{b}{2}\right)^2 + \left(c/2 - c/2\right)^2}$

$= \sqrt{\left(\frac{a}{2}\right)^2 + 0^2} = \sqrt{\frac{a^2}{4}} = \frac{a}{2}$

$\therefore \overline{MN}$'s length is half of $\overline{BC}$'s length.

Coordinate Geometry Proofs
SOLUTIONS

10. Let the coordinates of the rhombus be as shown:



Slope of $\overline{AC} = \frac{c-0}{a+b-0} = \frac{c}{a+b}$

Slope of $\overline{DB} = \frac{c-0}{b-a} = \frac{c}{b-a}$

We must show that

$\left(\frac{c}{a+b}\right)\left(\frac{c}{b-a}\right) = -1$

Since ABCD is a rhombus we know that $AB = BC$

$\therefore \sqrt{(a-0)^2+(0-0)^2} = \sqrt{(a+b-a)^2+(c-0)^2}$

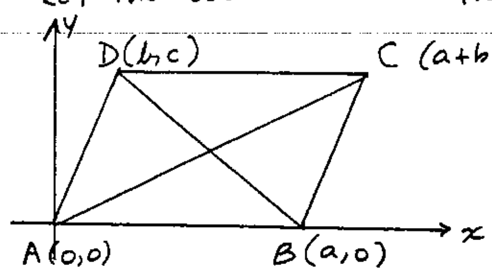
$\therefore a^2 = b^2 + c^2$

$\therefore a^2 - b^2 = c^2$

Then $\left(\frac{c}{a+b}\right)\left(\frac{c}{b-a}\right) = \frac{c^2}{b^2-a^2} = \frac{a^2-b^2}{b^2-a^2} = -1$

$\therefore$ they ($\overline{AC}$ and $\overline{DB}$) are perpendicular.

11. Let the coordinates of the parallelogram be as shown.



We know that $AC = DB$

$\therefore \sqrt{(a+b-0)^2+(c-0)^2} = \sqrt{(b-a)^2+(c-0)^2}$

$\therefore (a+b)^2+c^2 = (b-a)^2+c^2$

$\therefore a^2+2ab+b^2+c^2 = b^2-2ab+a^2+c^2$

$\therefore 2ab = -2ab$

$\therefore 4ab = 0$

$\therefore a = 0$ or $b = 0$

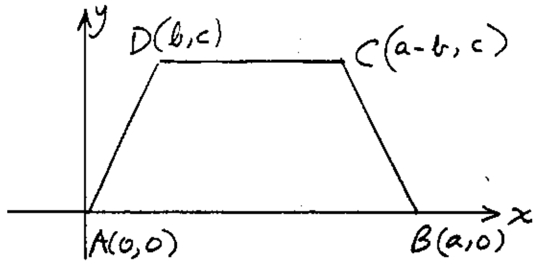
a can't be 0 because then B would coincide with A and the figure could not be a parallelogram. (there is no figure)

$\therefore$ b must be 0. This means D lies on the y axis. $\therefore \angle DAB = 90°$.

$\therefore$ ABCD is a rectangle since a parallelogram having 1 right $\angle$ is a rectangle.

Coordinate Geometry Proofs
SOLUTIONS

12. Let the coordinates of the trapezoid (isosceles) be as shown.



$AC = \sqrt{(a-b-0)^2 + (c-0)^2}$

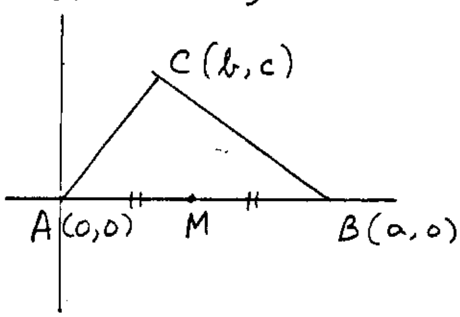
$= \sqrt{a^2 - 2ab + b^2 + c^2}$

$BD = \sqrt{(b-a)^2 + (c-0)^2}$

$= \sqrt{b^2 - 2ab + a^2 + c^2}$

$\therefore AC = BD$

13. Let the triangle have vertices as shown.



We need to show $\overline{AC} \perp \overline{BC}$. We will try to show that the product of the slopes of $\overline{AC}$ and $\overline{BC}$ is -1.

Slope of $\overline{AC} = \dfrac{c-0}{b-0} = \dfrac{c}{b}$

Slope of $\overline{BC} = \dfrac{c-0}{b-a} = \dfrac{c}{b-a}$

$\therefore$ (Slope of $\overline{AC}$)(Slope of $\overline{BC}$) $= \left(\dfrac{c}{b}\right)\left(\dfrac{c}{b-a}\right)$

$= \dfrac{c^2}{b(b-a)}$ ①

M, the midpoint of $\overline{AB}$, has coordinates of $\left(\frac{a}{2}, 0\right)$

Since $AM = MC$, then $\sqrt{(a/2 - 0)^2 + (0-0)^2} = \sqrt{(b - \frac{a}{2})^2 + (c-0)^2}$

$\therefore \quad \dfrac{a^2}{4} = \left(b - \dfrac{a}{2}\right)^2 + c^2$

$\dfrac{a^2}{4} = b^2 - ab + \dfrac{a^2}{4} + c^2$

$0 = b^2 - ab + c^2$

$\therefore \; ab - b^2 = c^2$ ②

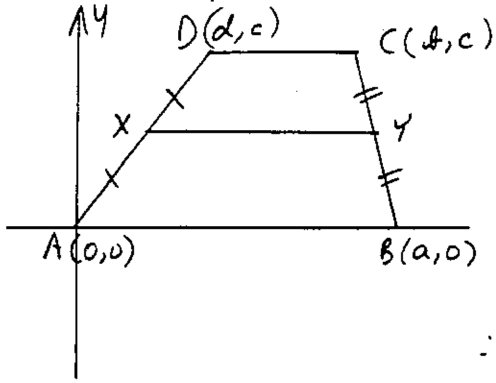
Going back to ① we have $\dfrac{c^2}{b(b-a)} = \dfrac{ab - b^2}{b(b-a)}$ (from ②) $= \dfrac{b(a-b)}{b(b-a)}$

$= -1.$

$\therefore \; \triangle ABC$ is a right $\triangle$.

Coordinate Geometry Proofs
SOLUTIONS

14. Let the trapezoid have coordinates as shown.



X has coordinates $\left(\frac{d+0}{2}, \frac{c+0}{2}\right) = \left(\frac{d}{2}, \frac{c}{2}\right)$

Y has coordinates $\left(\frac{b+a}{2}, \frac{c+0}{2}\right) = \left(\frac{b+a}{2}, \frac{c}{2}\right)$

$\therefore$ slope of $\overline{XY} = \dfrac{c/2 - c/2}{\frac{b+a}{2} - \frac{d}{2}} = \dfrac{0}{\frac{b+a-d}{2}}$

$= 0$

$\therefore \overline{XY} \parallel \overline{AB}$ since $\overline{AB}$'s slope is also 0.

$AB = \sqrt{(a-0)^2 + (0-0)^2} = a$

$XY = \sqrt{\left(\frac{b+a}{2} - \frac{d}{2}\right)^2 + \left(c/2 - c/2\right)^2} = \frac{b+a}{2} - \frac{d}{2} = \frac{b+a-d}{2}$

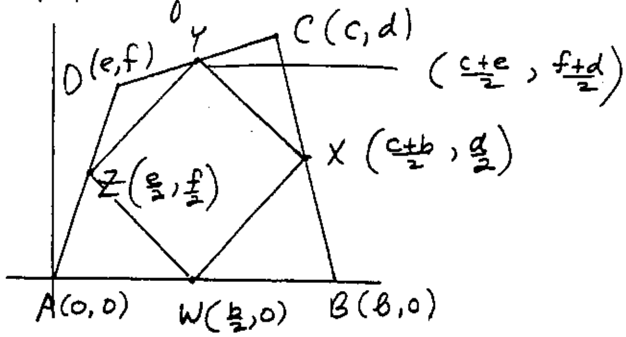
$DC = \sqrt{(b-d)^2 + (c-c)^2} = b - d$

$\frac{1}{2}(AB + DC) = \frac{1}{2}\left(a + b - d\right) = \frac{a+b-d}{2}$

$\therefore$

* Subtract coordinates so differences remain positive. Otherwise use absolute values when you take the $\sqrt{\ }$.

15. Let the quad. have coordinates as shown:



Slope of $\overline{YX} = \dfrac{\frac{f+d}{2} - \frac{d}{2}}{\frac{c+e}{2} - \frac{c+b}{2}}$

$= \dfrac{\frac{f}{2}}{\frac{e}{2} - \frac{b}{2}} = \dfrac{f}{e-b}$

Slope of $\overline{ZW} = \dfrac{\frac{f}{2} - 0}{\frac{e}{2} - \frac{b}{2}} = \dfrac{\frac{f}{2}}{\frac{e}{2} - \frac{b}{2}}$

$= \dfrac{f}{e-b}$

$\therefore \overline{YX} \parallel \overline{ZW}$. Similarly one can show that the slope of $\overline{YX}$ and $\overline{XW}$ are both $\frac{d}{c}$ and are $\therefore \parallel$. $\therefore$ Quad XYZW is a llgm.

Proofs By Mathematical Induction

SOLUTIONS

1\.

Is the statement true when n is say 3?

Basis step

Does $1+2+3 \overset{?}{=} \frac{3(3+1)}{2}$

$6 \overset{?}{=} \frac{3(4)}{2}$

Yes

Assume the statement is true for $n=k$

ie assume that $1+2+3+4+\cdots+k = \frac{k(k+1)}{2}$

Now prove that the statement is true when $n=k+1$.

That is, prove that

$$\underbrace{1+2+3+4+\cdots+k}_{\text{assumption}} + k+1 = \frac{(k+1)(k+1+1)}{2}$$

L.S.	R.S.
$= \frac{k(k+1)}{2} + k+1$	$\frac{(k+1)(k+2)}{2}$
$= \frac{k(k+1)+2(k+1)}{2}$	
$= \frac{(k+1)(k+2)}{2}$	

Since L.S. = R.S. we have proven the statement is true when $n=k+1$. ∴ the statement is true for all natural numbers.

2\. Is the statement true when $n=3$?

Basis Step

Does $2+6+10 = 2(3)^2$?

$18 = 2(9)$?

$18 = 18$ yes.

Assume that the statement is true for $n=k$

ie assume that $2+6+10+14+\cdots+4k-2 = 2k^2$

Now prove that the statement is true for $n=k+1$.

That is, show that

$$\underbrace{2+6+10+14+\cdots+4k-2}+4(k+1)-2 = 2(k+1)^2$$

L.S. $= 2k^2+4k+4-2$

$= 2k^2+4k+2$

$= 2(k^2+2k+1)$

$= 2(k+1)^2$

∴ true for all natural numbers

Proofs By Mathematical Induction

SOLUTIONS

3. Is the statement true for say $n = 3$?

Does $1 + 2 + 4 = 2^3 - 1$?

$7 = 7$ yes.

Assume the statement is true for $n = k$.

ie assume that $1 + 2 + 4 + 8 + \cdots + 2^{k-1} = 2^k - 1$

Now prove that it is true for $n = k + 1$.

ie Prove that $1 + 2 + 4 + 8 + \cdots + 2^{k-1} + 2^{k+1-1} = 2^{k+1} - 1$

$$= 2^k - 1 + 2^k$$
$$= 2^k + 2^k - 1$$
$$= 2(2^k) - 1$$
$$= 2^{k+1} - 1$$

∴ the statement is true for all natural numbers.

4. Is the statement true when $n = 3$?

Does $1^2 + 2^2 + 3^2 = \frac{3(3+1)(2(3)+1)}{6}$?

$$1 + 4 + 9 \stackrel{?}{=} \frac{3(4)(7)}{6}$$
$$14 \stackrel{?}{=} \frac{84}{6}$$
$$14 = 14 \checkmark$$

Assume that the statement is true when $n = k$.

ie assume that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Prove that the statement is true when $n = k + 1$

ie prove that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$$\frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$
$$= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6}$$
$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

∴ L.S. = R.S.

∴ true for all natural numbers.

Proofs By Mathematical Induction
SOLUTIONS

5. Is the statement true when $n=3$?

Does $1^2+3^2+5^2 = \frac{3(2(3)+1)(2(3)-1)}{3}$?

$$1+9+25 \stackrel{?}{=} \frac{3(7)(5)}{3} \ ?$$

$$35 = 35 \checkmark$$

Assume that the statement is true when $n=k$,
ie assume that $1^2+3^2+5^2+\cdots+(2k-1)^2 = \frac{k(2k+1)(2k-1)}{3}$

Prove that the statement is true when $n=k+1$.
ie Prove that $\underbrace{1^2+3^2+5^2+\cdots+(2k-1)^2}+(2(k+1)-1)^2 = \frac{(k+1)(2(k+1)+1)(2(k+1)-1)}{3}$

$$= \frac{(k+1)(2k+3)(2k+1)}{3}$$

$$= \frac{k(2k+1)(2k-1)}{3} + (2k+1)^2$$

$$= \frac{k(2k+1)(2k-1)+3(2k+1)^2}{3}$$

$$= \frac{(2k+1)\left(k(2k-1)+3(2k+1)\right)}{3}$$

$$= \frac{(2k+1)(2k^2-k+6k+3)}{3}$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3} = \frac{(2k+1)(2k+3)(k+1)}{3}$$

∴ it is true for all natural numbers.

6. Is the statement true when $n=4$?

Is 4^2+4 ÷ by 2 ?

Is 20 ÷ by 2 ? yes.

Assume that the statement is true when $n=k$.
That is assume that k^2+k is divisible by 2.
Prove that the statement is true for $n=k+1$.
That is, prove that $(k+1)^2+(k+1)$ is divisible by 2

$$(k+1)^2+k+1 = k^2+2k+1+k+1 = k^2+k+2k+2$$

$$= \underbrace{k^2+k}_{\text{divisible by 2 by assumption}} + \underbrace{2(k+1)}_{\text{divisible by 2 by form}}$$

∴ sum is divisible by 2.

∴ true for all natural numbers.

Proofs By Mathematical Induction
SOLUTIONS

7. Is the statement true when $n=4$?
<u>ie</u> is $4^3+2(4)$ divisible by 3?
$64+8=72$ which is divisible by 3.

Assume that the statement is true when $n=k$.
ie assume that k^3+2k is divisible by 3.
Prove that the statement is true for $n=k+1$.
<u>ie</u> prove that $(k+1)^3+2(k+1)$ is divisible by 3.

$$= k^3+3k^2+3k+1+2k+2$$
$$= k^3+2k+3k^2+3k+1+2$$
$$= k^3+2k+3k^2+3k+3$$
$$= \underbrace{k^3+2k}_{\text{a multiple of 3 by assumption}} + \underbrace{3(k^2+k+1)}_{\text{a multiple of 3}}$$

$\therefore$ sum is a multiple of 3.
$\therefore$ true for all natural numbers.

8. Is the statement true for $n=5$?
Is $5(5^2+5)$ divisible by 6?
Is $5(30)$ divisible by 6?
Is 150 divisible by 6? <u>YES</u>

Assume that the statement is true for $n=k$.
<u>ie</u> assume that $k(k^2+5)$ is divisible by 6
Now show that the statement is true for $n=k+1$
<u>ie</u> show that $(k+1)((k+1)^2+5)$ is divisible by 6

$$= (k+1)(k^2+2k+1+5)$$
$$= (k+1)(k^2+5+2k+1)$$
$$= k(k^2+5)+k(2k+1)+1(k^2+5)+1(2k+1)$$
$$= k(k^2+5)+2k^2+k+k^2+5+2k+1$$
$$= k(k^2+5)+3k^2+3k+6$$
$$= k(k^2+5)+3(k^2+k+2)$$
$$= \underbrace{k(k^2+5)}_{\text{a multiple of 6 by assumption}} + \underbrace{3(k(k+1)+2)}_{\text{a multiple of 3}}$$

a multiple of 3 by form, but since k and $k+1$ are consecutive integers, their product will be even. An even number + 2 is still even. $\therefore k(k+1)+2$ is even.
$\therefore 3(k(k+1)+2)$ is also a multiple of 6.

$\therefore$ their sum will be a multiple of 6.

Proofs By Mathematical Induction

SOLUTIONS

(9) Is the statement true when $n = 2$?
Is $2 < 2+1$? Yes.
Assume that the statement is true when $n=k$.
that is, assume that $k < k+1$.
Is the statement true when $n=k+1$?
ie Prove that $k+1 < k+1+1$ ie $(k+2)$
Since $k < k+1$ (by assumption)
and $1 = 1$
then $k+1 < k+1+1$ (+n of 1 to both sides).
$\therefore k+1 < k+2$ $\therefore$ true when $n=k+1$
$\therefore$ true for all natural numbers.

(10) Is the statement true when $n=3$?
is $2 \leq 2^3$? Yes.
Assume that the statement is true for $n=k$.
ie assume that $2 \leq 2^k$.
Now prove that the statement is true for $n=k+1$.
ie Prove that $2 \leq 2^{k+1}$
If $2 \leq 2^k$ (assumption)
then $2 \cdot 2 \leq 2 \cdot 2^k$ (mult. by 2)
$\therefore$ $4 \leq 2^{k+1}$
but $2 \leq 4$
$\therefore$ $2 \leq 2^{k+1}$
$\therefore$ the statement is true for $n=k+1$.
$\therefore$ true for all natural numbers.

Proofs By Mathematical Induction

SOLUTIONS

11. Is the statement true when $n=6$?

ie is $2^6 > 6^2$?

$64 > 36$ yes ✓

Assume that the statement is true for $n=k$.

ie assume that $2^k > k^2$.

Now prove that the statement is true for $n=k+1$.

ie prove that $2^{k+1} > (k+1)^2$ ie prove $2^{k+1} > k^2 + 2k + 1$ ③

If $2^k > k^2$ ① (assumption)

then $2^k + 2^k > 2^k + k^2$ (addition of 2^k to both sides of ①)

$\therefore 2(2^k) > 2^k + k^2$

$\therefore 2^{k+1} > 2^k + k^2$ ②

Now $2^k + k^2$ will be $> k^2 + 2k+1$ if 2^k is $> 2k+1$. This is true if $k = 2$ and so will certainly be true for any n value since $n \geq 5$.

$\therefore$ since $2^{k+1} > 2^k + k^2$ ② AND $2^k + k^2 > k^2 + 2k + 1$,

then $2^{k+1} > k^2 + 2k + 1$ which is what we need to prove in ③.

$\therefore$ true for all natural numbers.

12. Is the statement true when $n = 3$?

Is $2(3) \leq 2^3$?

$6 \leq 8$ yes ✓

Assume that the statement is true when $n=k$.

ie assume that $2k \leq 2^k$.

Now prove that the statement is true when $n=k+1$.

That is, prove that $2(k+1) \leq 2^{k+1}$. ③

Since $2k \leq 2^k$ (assumption)

then $2(2k) \leq 2(2^k)$ (mult. by 2)

$4k \leq 2^{k+1}$ ①

How does $4k$ compare to $2(k+1)$

$4k - 2(k+1) = 2k - 1$ which is positive since $k \geq 1$.

$\therefore$ if $4k - 2(k+1)$ is positive then $2(k+1) < 4k$ ②

Combining ① and ② we have

$2(k+1) \leq 2^{k+1}$ which proves ③

$\therefore$ true for all natural numbers.

Proofs By Mathematical Induction
SOLUTIONS

13. Is the statement true when $n = 2$?
Is $x^2 - y^2$ is divisible by $x - y$?
Since $(x-y)(x+y) = x^2 - y^2$, then $x^2 - y^2$ is divisible by $x - y$.

Assume that the statement is true when $n = k$.
ie assume that $x^k - y^k$ is divisible by $x - y$.
Now prove that the statement is true when $n = k+1$.
ie prove that $x^{k+1} - y^{k+1}$ is divisible by $x - y$.

Now $x^{k+1} - y^{k+1}$

$$= x^{k+1} \underbrace{- xy^k + xy^k}_{\text{really } 0} - y^{k+1}$$

$$= x\underbrace{(x^k - y^k)}_{\substack{\text{divisible}\\ \text{by } x-y \\ \text{by assumption}}} + \underbrace{y^k(x-y)}_{\substack{\text{divisible by}\\ x-y \text{ since } x-y\\ \text{is a factor}}}$$

Since each term is divisible by $x - y$ so is their sum.
$\therefore$ statement is true when $n = k+1$. $\therefore$ true for all natural numbers.

14. Is the statement true when $n = 3$?
Does $a + (a+d) + (a+2d) = \frac{3}{2}\left[2a + (3-1)d\right]$?
Does $3a + 3d = \frac{3}{2}\left[2a + 2d\right]$?
$3a + 3d = 3(a+d)$ yes. ✓

Assume that the statement is true when $n = k$.
ie assume that $a + (a+d) + (a+2d) + \cdots + (a + (k-1)d) = \frac{k}{2}\left[2a + (k-1)d\right]$
Prove that the statement is true when $n = k+1$.
ie prove that $\underbrace{a + (a+d) + (a+2d) + \cdots + (a + (k-1)d)} + (a + (k+1-1)d) =$

$$= \frac{k}{2}\left[2a + (k-1)d\right] + (a + (k+1-1)d)$$
$$= \frac{k}{2}\left[2a + (k-1)d\right] + \frac{2(a+kd)}{2}$$
$$= \frac{1}{2}\left\{k\left[2a + (k-1)d\right] + 2(a+kd)\right\}$$
$$= \frac{1}{2}\left\{k\left[2a + kd - d\right] + 2a + 2kd\right\}$$
$$= \frac{1}{2}\left[2ka + k^2d - kd + 2a + 2kd\right]$$
$$= \frac{1}{2}\left[2ka + k^2d + 2a + 2kd\right]$$

$$\frac{k+1}{2}\left[2a + (k+1-1)d\right]$$
$$\frac{k+1}{2}\left[2a + kd\right]$$
$$\frac{1}{2}\left[(k+1)(2a+kd)\right]$$
$$\frac{1}{2}\left[2ak + k^2d + 2a + kd\right]$$

$\therefore$ true for all natural numbers

Proofs By Mathematical Induction

SOLUTIONS

15. Is the statement true when $n = 3$?
ie does a Δ have interior $\angle$'s whose sum is $(3-2)180$ or 180? Yes.

Assume that the statement is true when $n=k$.
ie assume that a convex polygon of k sides has an interior angle sum of $(k-2)180$.

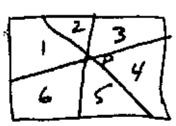
Is it true for $n=k+1$
ie Prove that the interior angles of a convex polygon of $(k+1)$ sides have a sum of $(k+1-2)\,180°$. ①

A polygon of k sides permits one to draw $k-3$ interior diagonals from a particular vertex, say A because connecting A to itself or to its adjacent vertices (one per side) does not create interior diagonals. By drawing $k-3$ diagonals the polygon is divided into $k-2$ Δ's each of whose sums is $180°$ hence giving the $(k-2)\,180°$ total. When the $k+1$ st vertex is created to give the $k+1$st side then A can be connected to one more vertex resulting in the creation of one more Δ whose interior $\angle$'s have a sum of $180°$. $\therefore$ a polygon of $k+1$ sides will have an interior $\angle$ sum of

$$\underbrace{(k-2)180°}_{\text{sum for a } k \text{ sides}} + \underbrace{180°}_{\text{increase for } k+1\text{st side}} = \underline{180(k-2+1)} \quad ②$$

Since ② = ① the statement is true when $n=k+1$. $\therefore$ true for all natural numbers (≥ 3).

16. Is the statement true when $n = 3$. ie do 3 lines drawn through a given point divide the plane into 6 regions?



Yes.

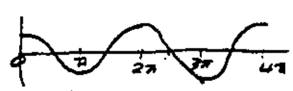
Assume the statement is true when $n=k$. ie assume that k lines that pass through P will divide the plane into $2k$ regions.
Prove that the statement is true when $n=k+1$. ie prove that the plane will be divided into $2(k+1)$ regions by $k+1$ lines.
Since k lines divide the plane into $2k$ regions, when 1 more line is drawn through P it will cut through two existing regions creating 2 more regions (each of those 2 existing regions is cut into 2 making 4 regions where before there were 2). $\therefore$ there are

now $2k+2$ regions or $2(k+1)$ regions which is what we had to show. $\therefore$ the statement is true when $n=k+1$. $\therefore$ it is true for all natural numbers.

17. Is the statement true when $n=5$. Can a postage of 5¢ be made up using two cent and three cent stamps? Yes $5¢ = 1(2¢)+1(3¢)$
Assume that the statement is true when $n=k$. i.e. assume that a postage of k¢ can be made up using two cent and three cent stamps.
Show that the statement is true when $n=k+1$. i.e. show that a postage of $k+1$ cents can be made up of two and three cent stamps.

If k is odd, then $k+1$ will be even. $\therefore$ $k+1$¢ can be made using $\frac{1}{2}(k+1)$ two cent stamps. If k is even then $k+1$ will be odd and $k-2$ will be even. The $k-2$¢ can be made up of $\frac{1}{2}(k-2)$ two cent stamps by using 1 three cent stamp in addition to the $\frac{1}{2}(k-2)$ two cent stamps you will make up your postage of $k+1$ cents. $\therefore$ the statement is true when $n=k+1$. $\therefore$ it is true for all natural numbers.

18. Is the statement true when $n=3$? Does $\cos 3\pi = (-1)^3$? Yes
$\cos 3\pi = -1 = (-1)^3$.



Assume that the statement is true when $n=k$. i.e. assume that $\cos k\pi = (-1)^k$. Prove that the statement is true when $n=k+1$.
i.e. prove $\cos(k+1)\pi = (-1)^{k+1}$

$$\begin{aligned}\text{Now } \cos(k+1)\pi = \cos(k\pi+\pi) &= \cos k\pi \cos\pi - \sin k\pi \sin\pi \\ &= (-1)^k(-1) - (\sin k\pi)(0) \\ &= (-1)^{k+1} - 0 \\ &= (-1)^{k+1}\end{aligned}$$

$\therefore$ statement is true when $n=k+1$. $\therefore$ it is true for all natural numbers.